

Relative-locality distant observers and the phenomenology of momentum-space geometry

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We study the translational invariance of the relative-locality framework proposed in arXiv:1101.0931, which had been previously established only for the case of a single interaction. We provide an explicit example of boundary conditions at endpoints of worldlines, which indeed ensures the desired translational invariance for processes involving several interactions, even when some of the interactions are causally connected (particle exchange). We illustrate the properties of the associated relativistic description of distant observers within the example of a κ -Poincaré-inspired momentum-space geometry, with de Sitter metric and parallel transport governed by a non-metric and torsionful connection. We find that in such a theory simultaneously-emitted massless particles do not reach simultaneously a distant detector, as expected in light of the findings of arXiv:1103.5626 on the implications of non-metric connections. We also show that the theory admits a free-particle limit, where the relative-locality results of arXiv:1102.4637 are reproduced. We establish that the torsion of the κ -Poincaré connection introduces a small (but observably-large) dependence of the time of detection, for simultaneously-emitted particles, on some properties of the interactions producing the particles at the source.

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I. INTRODUCTION AND SUMMARY

The relative-locality framework of Refs. [1, 2] is centered on the possibility of a non-trivial geometry for momentum space, and links to those geometric properties some effects of relativity of spacetime locality, such that events established to be coincident by nearby observers are not described as coincident in the coordinatization of spacetime by distant observers. Interestingly, just like the relativity of simultaneity implies that there is no observer-independent projection from spacetime to separately space and time, relative locality implies that there is no observer-independent projection from a one-particle phase space to a description of the particle separately in spacetime and in momentum space. Besides its intrinsic interest from the point of view of relativity research, this relative-locality framework appears to be also relevant [1, 2] for the understanding of several issues which have emerged in the recent quantum-gravity literature. Indeed several approaches to the study of the quantum-gravity problem have led to speculations about nonlinearities in momentum space that may admit geometric description (see, *e.g.*, Refs. [3–8] and references therein), and some related studies had hinted at possibly striking implications of such nonlinearities for the fate of locality at the Planck scale [9–14].

Evidently a pivotal role for the success (or failure) of this relative-locality proposal will be played by investigations of the relativistic description of distant observers, which is the main focus of the study we are here reporting. This is in itself an aspect of relative locality which is rather intriguing from a conceptual perspective. In previous evolutions of our relativistic theories the most subtle issues always concerned boost transformations, and therefore the relativistic description of pairs of observers with a relative boost. Translational invariance, and therefore the relativistic description of distant observers, always admitted an elementary and fully intuitive description. This is particularly clear when looking at the transition from Galilean relativity, and its description of relative rest, to special relativity, with Einstein’s description of relative simultaneity and rest: grasping the physical content of special relativity proved challenging because of properties of special-relativistic boosts, which force us to abandon a “common sense understanding”. But special relativistic translations are no less trivial than Galilean translations. This is because the special-relativistic notion of relative simultaneity is already fully characterized when focusing on pairs of observers connected by a pure boost transformation. But the relative-locality notion that an interaction established to be local by nearby observers may be described as nonlocal by distant observers evidently implies a crucial role for the identification of a corresponding formalization of translational symmetries, and we must therefore expect that ensuring a relativistic description of distant observers should be one of the main challenges for the formalization of relative locality.

A crucial step toward the understanding of these issues here of interest, concerning the interplay between relative locality and translation symmetries, is already found in Refs. [1, 2]. For example, according to Ref. [1], one could describe the process in Figure 1, which is the idealized case with 3 particles of energy-momenta k_μ , p_μ , q_μ all incoming, on the basis of the following action:

$$S^{example} = \int_{-\infty}^{s_0} ds (x^\mu \dot{k}_\mu + y^\mu \dot{p}_\mu + z^\mu \dot{q}_\mu + \mathcal{N}_k C[k] + \mathcal{N}_p C[p] + \mathcal{N}_q C[q]) - \xi^\mu \mathcal{A}_\mu(s_0). \quad (1)$$

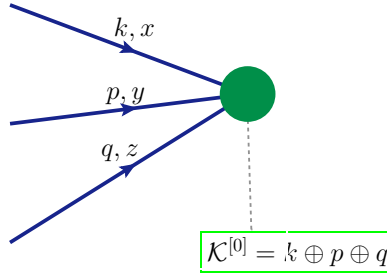


Figure 1. We show here a simple example of process for which the observations reported in Ref. [1] suffice for establishing translational invariance (with relative locality). Specifically the graph intends to describe the idealized case of a process with 3 particles of energy-momenta k_μ , p_μ , q_μ , all incoming. Here and elsewhere in this manuscript we describe pictorially some processes using a graphical scheme which is mainly of evocative valence. The lines in the graph are not intended as representatives of worldlines of particles or other fine aspects of the evolution of variables in terms of the affine parameter. They should rather be looked at, going from left to right, as a schematic portrait of the discrete steps in the redistribution of momentum among particles, changing at every subsequent interaction (but the case in this figure is a single-interaction process). A similar graphical characterization of processes is often adopted for quantum-field-theory Feynman diagrams (but our entire analysis is confined to the context of classical particles).

As we shall here discuss in greater detail in Section V, the bulk part of $S^{example}$ ends up characterizing [1] the propagation of the 3 particles, with the Lagrange multiplier \mathcal{N}_k (and similarly \mathcal{N}_p and \mathcal{N}_q) enforcing the on-shell relation $C[k] = D^2(k) - m^2$, with $D^2(k)$ in turn derived from the metric on momentum space as the distance of k_μ from the origin of momentum space. And

the form of the boundary term $\xi^\mu \mathcal{K}_\mu(s_0)$ is such that [1] the Lagrange multipliers ξ^μ enforce the condition $\mathcal{K}_\mu(s_0) = 0$, so that by taking for \mathcal{K}_μ a suitable composition of the momenta k_μ, p_μ, q_μ the boundary terms enforces a law of conservation of momentum at the interaction. The form of the law of composition of momenta used for the conservation law $\mathcal{K}_\mu(s_0) = 0$ is governed by the affine connection on momentum space [1], and may involve nonlinear terms which are ultimately responsible for the relativity of spacetime locality. This is indeed seen by studying the invariance of the action $\mathcal{S}^{example}$ under translations of the coordinates of worldline points $x^\mu(s), y^\mu(s), z^\mu(s)$, which each observer introduces as variables that are canonically conjugate to the coordinates on momentum space $k_\mu(s), p_\mu(s), q_\mu(s)$.

The observations used in Ref. [1] in the derivations that established the presence of this translational invariance appeared to rely crucially on some simplifications afforded only by the case of a single-interaction process. As here stressed in Section V, for a single interaction several alternative choices of boundary terms enforcing the same conservation law are consistent with the presence of relativistic translation symmetries within the relative-locality framework of Ref. [1]. However, the demands of translational invariance become much more constraining when 2 or more interactions are causally connected, *i.e.* when a particle outgoing from one interaction is incoming into another interaction. In Section VII we provide an explicit example of formulation of the boundary terms which ensures translational invariance when several causally-linked interactions are analyzed. And the logical structure of our proposal is easily described: translation transformations are generated by the total-momentum charge (obtained from individual particle momenta via the connection-induced composition law) and the boundary terms are written as differences between the total momentum before the interaction and after the interaction (so that the associated constraints automatically ensure conservation of the total momentum).

Before getting to that main part of our analysis it will be useful to do some preparatory work. In the next section we motivate our focus on results that are obtained only at leading order in the deformation scale, by observing that, if indeed the deformation scale is roughly given by the Planck scale, the experimental sensitivities foreseeable at least for the near future will not afford us investigations going much beyond the leading-order structure of the geometry of momentum space. And we also show that working at leading order not only simplifies matters in the way that is commonly encountered in physics, by shortening some computations, but in this case also provides some qualitative simplifications, including most notably the fact that at leading order the momentum-composition laws of the relative-locality framework are automatically associative.

Then in Sections III and IV we characterize the specific example of relative-locality momentum space on which we shall test our proposal for relativistic translational symmetries. This is based on results obtained in the κ -Minkowski/ κ -Poincaré framework, where indeed evidence of a deformed on-shell relation and of a nonlinear law of composition of momenta has been discussed for more than a decade, but without appreciating the implications for how distant observers would characterize events that are found to be coincident by nearby observers. We find that this κ -Poincaré-inspired momentum space has de Sitter metric and parallel transport governed by a non-metric and torsionful connection.

Equipped with these preliminary observations we then discuss, in Section V, the challenges that must be dealt with in seeking a translationally-invariant description of chains of causally connected interactions within the relative-locality framework proposed in Ref. [1]. The main insight gained from the analysis reported in Section V is that in order to achieve translational invariance it is not sufficient to ensure that the boundary terms at endpoints of worldlines enforce some suitable momentum-conservation laws, since in general two such choices of boundary terms at the endpoints of a finite worldline (a worldline going from one interaction to another) will spoil translational invariance.

Section VI is a short aside on some characterizations of relative locality confined to a Hamiltonian description of free particles [15–18] which serves two purposes: it prepares the rest of our analysis by reviewing some concepts about the type of symplectic structure that is traditionally used in the κ -Minkowski/ κ -Poincaré literature, and its characterization of relative locality for free particles is then of reference for our description of a free-particle limit, which is an important corollary result of our proposal for interacting particles.

It is in Section VII that the reader finds our main results concerning the proposal and analysis of a relativistic formulation of processes involving several interactions, within the general framework of Ref. [1], with translational invariance assured by a corresponding specification of the boundary conditions that implement momentum conservation. We test the robustness of our proposal mainly by applying it to the illustrative example of the κ -Poincaré-inspired momentum space.

Section VIII contains our results that are of particular significance from the perspective of phenomenology. We show that, within our setup for κ -Poincaré interacting particles, simultaneously-emitted massless particles do not necessarily reach the same detector at the same time. Since our κ -Poincaré momentum space has nonmetricity, this is consistent with the thesis put forward in Ref. [19], according to which these time delays at detection for simultaneously-emitted massless particles are to be expected when nonmetricity is present. In addition we also investigate how the torsion of our κ -Poincaré momentum space affects these time-of-detection delays, an issue for which no previous result is applicable. And we find that the torsion does affect the time delays, by essentially rendering the effect non-systematic: the time-of-detection difference for two simultaneously emitted massless particles depends non only on the momenta of the two particles involved but also on some properties of the events that emitted the two particles. We also discuss the first elements of a phenomenology that could exploit this striking feature.

In deriving the results reported in Sections VII and VIII we use κ -Poincaré illustrative example non only in the sense that we adopt the metric and connection of the “ κ -momentum space” (of Sections III and IV) but also by imposing upon us the use of the nontrivial symplectic structure that is preferred in the κ -Poincaré literature. However, in Section IX we keep the κ -Poincaré

momentum space while switching to a trivial symplectic structure, and we reproduce again the results of Section VIII. This allows us to establish that the predictions derived in Section VIII are purely manifestations of momentum space geometry.

Section X contains some closing remarks, mostly focusing on the outlook of the relative-locality research program.

We adopt units such that the speed-of-light scale (speed of massless particles in the infrared limit) and the reduced Planck constant are 1 ($c = 1 = \hbar$). And we denote by ℓ the momentum-space-deformation scale. Of course, ℓ carries dimensions of inverse momentum, and a natural quantum-gravity-inspired estimate would be to have $|\ell^{-1}|$ roughly of the order of the Planck scale. The issues studied in this manuscript are of exactly the same nature in the case of a 4D momentum space and in the case of a 2D momentum space, and we shall often (but not always) focus for definiteness and simplicity on the 2D case. When not otherwise specified we shall switch between 4D and 2D formulas by simply denoting with p_0, p_j the momentum in the 4D case and with p_0, p_1 the momentum in the 2D case.

II. LEADING-ORDER ANATOMY OF RELATIVE-LOCALITY MOMENTUM SPACES

Refs. [1, 2] (also see Refs. [19, 20]) raised the issue of determining experimentally the geometry of momentum space, much like it is traditional in physics to study experimentally the geometry of spacetime.

It is however important to notice a crucial difference: while we do have experimental access to distance scales larger than the scales of curvature of spacetime, it is very unlikely that in the foreseeable future we could have experimental access to momentum scales even just comparable to the Planck scale, which is the natural candidate for the scale of curvature of the relative-locality momentum space [1].

It should be appreciated that this disappointing limitation of our horizons on the geometry of momentum space can also be turned in some sense into a powerful weapon for the phenomenology of momentum-space geometry: evidently all we need is a characterization of the geometry of momentum space near the origin, where $|p| \ll |\ell|^{-1} \simeq M_p$. And at least at first this will essentially be focused on the search of leading-order evidence of a nontrivial geometry of momentum space.

For what concerns the affine connection on momentum space, responsible for the nontrivial properties of the law of composition of momenta [1], all we need for the purposes of this phenomenology are the (ℓ -rescaled) connection coefficients on momentum space evaluated at $p_\mu = 0$, which we denote by $\Gamma_\mu^{\alpha\beta}$:

$$(p \oplus q)_\mu \simeq p_\mu + q_\mu - \ell \Gamma_\mu^{\alpha\beta} p_\alpha q_\beta + \dots$$

And evidently the fact that the phenomenology only needs leading-order results implies (also considering that we already rescaled the connection coefficients by the Planck scale) that we can treat the $\Gamma_\mu^{\alpha\beta}$ as pure numbers.

Analogous considerations lead us to focus on momentum-space metrics that are at most linear in the momenta:

$$g^{\mu\nu} = \eta^{\mu\nu} + \ell h^{\mu\nu\rho} p_\rho \quad (2)$$

and, just like the $\Gamma_\mu^{\alpha\beta}$, we should handle the coefficients $h^{\mu\nu\rho}$ as pure numbers in our leading-order phenomenology.

In this manuscript we shall mainly work only at leading order in the deformation scale, and it will be evident that this provides with significant advantages. In particular, at leading order in the deformation scale the momentum-composition law is always associative. This can be established by writing a general leading-order composition law as follows:

$$(k \oplus p)_\mu = k_\mu + p_\mu - \ell \Gamma_\mu^{\alpha\beta} k_\alpha p_\beta, \quad (3)$$

and then noticing that indeed (of course to leading-order accuracy)

$$[(k \oplus p) \oplus q]_\mu = k_\mu + p_\mu + q_\mu - \ell \Gamma_\mu^{\alpha\beta} (k_\alpha p_\beta + k_\alpha q_\beta + p_\alpha q_\beta) = [k \oplus (p \oplus q)]_\mu. \quad (4)$$

Beyond leading order the composition law could be nonassociative, and in that case one could appreciate the curvature of the momentum-space connection, with interesting but technically challenging consequences which we shall not encounter in this manuscript, and will never be encountered when working at leading order in the deformation scale.

The fact that our horizons on the geometry of momentum space probably are confined to leading order may be viewed as an unpleasant philosophical limitation, but pragmatically can be turned into a powerful asset for phenomenology work on relative-locality momentum spaces, since the task of phenomenologists then is very clearly and simply specified: the target should be to determine experimentally (as accurately as possible) a few dimensionless numbers for the leading-order (and possibly the next-to-leading order) geometry of momentum space.

To make this point fully explicit let us for simplicity imagine a by 2D relative-locality momentum space. In the 2D case a full leading-order characterization of the momentum-space geometry requires establishing experimentally (in hypothetical 2D

experiments) the 8 dimensionless parameters of the affine connection on momentum space,

$$\begin{array}{cccc} \Gamma_0^{00}, & \Gamma_0^{01}, & \Gamma_0^{10}, & \Gamma_0^{11}, \\ \Gamma_1^{00}, & \Gamma_1^{01}, & \Gamma_1^{10}, & \Gamma_1^{11} \end{array}$$

and the 6 dimensionless parameters of the leading-order description of the metric, which one can conveniently encode¹ into the 6 free parameters of the associated Christoffel symbols $C_{\alpha}^{\mu\nu}$,

$$\begin{array}{ccc} C_0^{00}, & C_0^{01} = C_0^{10}, & C_0^{11}, \\ C_1^{00}, & C_1^{01} = C_1^{10}, & C_1^{11}. \end{array}$$

III. SOME KNOWN PROPERTIES OF THE κ -POINCARÉ HOPF ALGEBRA AND κ -MINKOWSKI SPACETIME

A. κ -momentum space

In this section we describe the construction of the momentum space motivated by the κ -Poincaré framework [6, 21, 22], which we shall call here, for short, the “ κ -momentum space”. This κ -momentum space will provide for us an example of momentum space, of some independent interest, on which to illustrate in tangible way the efficacy of the characterization of relative-locality distant observers, which is our main objective for this manuscript.

In this subsection we follow Ref. [23] so we describe κ -momentum space as a manifold of the group $\text{AN}(3)$ (dubbed also the Borel group), which is, as a manifold, essentially a half of de Sitter space. The $\text{AN}(3)$ group is a subgroup of the de Sitter $\text{SO}(4, 1)$ group, defined by its Lie algebra $\mathfrak{an}(3)$ which has the following form:

$$[\mathcal{X}^0, \mathcal{X}^i] = -i\ell \mathcal{X}^i, \quad [\mathcal{X}^i, \mathcal{X}^j] = 0. \quad (6)$$

This algebra is a subalgebra of $\mathfrak{so}(4, 1)$ and one can represent it as an algebra of 5×5 real matrices, with the matrices representing \mathcal{X}^i being nilpotent. Knowing the form of the Lie algebra, one can readily write down a group element. It is convenient to split it into the product of two elements, one generated by nilpotent elements \mathcal{X}^i and the second generated by the abelian one \mathcal{X}^0 ,

$$\text{AN}(3) \ni g(p) = \exp(ip_i \mathcal{X}^i) \exp(ip_0 \mathcal{X}^0). \quad (7)$$

Clearly p_μ can be thought of as the coordinates on the group manifold.

Since $\text{AN}(3)$ is a subgroup of de Sitter group $\text{SO}(4, 1)$, $g(p)$ defined by (7) acts naturally on points of the five dimensional Minkowski space M^5 . Therefore, if we take a point O , the group $\text{AN}(3)$ as a manifold is just a set of all points of the form gO . If O has coordinates $(0, \dots, 0, 1/\ell)$ than the point $g(p)O$, with $g(p)$ given by (7) and represented as a 5×5 matrix has Minkowski coordinates

$$\begin{aligned} P_0(p_0, p_i) &= \frac{1}{\ell} \sinh \ell p_0 - \frac{\ell p_i^2}{2} e^{-\ell p_0}, \\ P_i(p_0, p_i) &= p_i e^{-\ell p_0}, \\ P_4(p_0, p_i) &= -\frac{1}{\ell} \cosh \ell p_0 + \frac{\ell p_i^2}{2} e^{-\ell p_0}. \end{aligned} \quad (8)$$

One can easily check by direct computation that the coordinates $P_I = (P_\mu, P_4)$, $\mu = 0, \dots, 3$ of these points satisfy the conditions

$$P_0^2 - P_1^2 - P_2^2 - P_3^2 - P_4^2 = -\frac{1}{\ell^2}, \quad (9)$$

and (assuming that ℓ is negative)

$$P_0 + P_4 > 0. \quad (10)$$

¹ We are here implicitly using the fact that our “leading-order momentum-space metrics” can be parametrized, in the example of the 2D case, equivalently in terms of the 6 independent numbers that specify $h^{\mu\nu\sigma}$ or in terms of the 6 independent Christoffel symbols. Indeed one finds that

$$C_p^{\mu\nu} = \frac{1}{2} g_{\rho\sigma} (g^{\sigma\mu, \nu} + g^{\nu\sigma, \mu} - g^{\mu\nu, \sigma}) = \frac{\ell}{2} \eta_{\rho\sigma} (h^{\sigma\mu\nu} + h^{\nu\sigma\mu} - h^{\mu\nu\sigma}) \quad (5)$$

Thus, as a manifold, the $\text{AN}(3)$ group is an open subset of the four dimensional de Sitter space (9) defined by the condition (10), and the points in this manifold can be parametrized by coordinates p_μ . It is worth noticing in passing that the unit element of the group $\text{AN}(3)$, $g(0)$ naturally corresponds to the zero momentum point $p_\mu = 0$ of the momentum space, whose existence is required for the relative locality construction [1].

Since our momentum space, $\text{AN}(3)$, is defined as a hyper-surface imbedded in the five dimensional Minkowski space it possesses a natural induced metric, which can be obtained by inserting the relations (8) into the five dimensional Minkowski metric

$$ds^2 = dP_0^2 - dP_1^2 - dP_2^2 - dP_3^2 - dP_4^2 .$$

Using (9) one finds that this metric is nothing but the de Sitter metric in flat coordinates

$$ds^2 = dp_0^2 - e^{-2\ell p_0} (dp_1^2 + dp_2^2 + dp_3^2) . \quad (11)$$

We shall use this form of the metric in the next section, in the derivation of the on-shell relation of a particle on the κ -momentum space.

Since our momentum space is a group manifold it is natural to assume that the momentum composition is defined by the group multiplication law. If we have two group elements $g(p)$ and $g(q)$ then their product is a group element itself so that we can define the momentum composition \oplus as follows:

$$g(p)g(q) = g(p \oplus q) . \quad (12)$$

It is worth stressing that since the group multiplication is associative, the composition \oplus is associative as well.

In the case of the $\text{AN}(3)$ group elements defined by (7) we find

$$g(p)g(q) = \exp \left(iX^i (p_i + e^{\ell p_0} q_i) \right) \exp \left(iX^0 (p_0 + q_0) \right) , \quad (13)$$

so that

$$(p \oplus q)_i = p_i + e^{\ell p_0} q_i , \quad (p \oplus q)_0 = p_0 + q_0 , \quad (14)$$

which to the leading order in ℓ reads

$$(p \oplus q)_i = p_i + q_i + \ell p_0 q_i + O(\ell^2) , \quad (p \oplus q)_0 = p_0 + q_0 . \quad (15)$$

We can then introduce $\ominus p$, the “antipode” of p , using the fact that the inverse of a group element is a group element itself:

$$g^{-1}(p) = g(\ominus p) , \quad g^{-1}(p)g(p) = 1 \Leftrightarrow p \oplus (\ominus p) = 0 \quad (16)$$

and in the case of the $\text{AN}(3)$ group we find

$$(\ominus p)_i = -e^{-\ell p_0} p_i , \quad (\ominus p)_0 = -p_0 \quad (17)$$

and in the leading order we have

$$(\ominus p)_i = -(1 - \ell p_0) p_i + O(\ell^2) , \quad (\ominus p)_0 = -p_0 . \quad (18)$$

In closing this subsection let us also observe that when the momentum space is a Lie group there is a natural way to construct a free particle action. The idea is to identify the position space with a linear space dual to the Lie algebra (as a vector space) and to make use of the canonical pairing between these dual spaces. Concretely let us define the basis of the vector space \mathcal{Y}_μ dual to the Lie algebra $\mathfrak{an}(3)$ as follows:

$$\langle \mathcal{Y}_\mu, X^\nu \rangle = \delta_\mu^\nu . \quad (19)$$

And let us take the space dual to the Lie algebra of $\text{AN}(3)$ to be the space of positions so that

$$x = x^\mu \mathcal{Y}_\mu . \quad (20)$$

Then the kinetic term of the action of a particle with $\text{AN}(3)$ momentum space is²

$$L^{kin} \equiv - \left\langle x, g^{-1} \frac{d}{d\tau} g \right\rangle . \quad (21)$$

² In the mathematical literature the symplectic form associated with this kinetic term is called Kirillov symplectic form.

Substituting (7), (19), and (20) into (21) one easily finds that

$$L^{kin} = x^\mu \dot{p}_\mu - \ell p_i x^i \dot{p}_0 . \quad (22)$$

It is worth noticing that the same procedure can be applied to the standard case with flat momentum space, when the group associated with momentum composition is just an abelian group \mathbb{R}^4 (in our case we get the abelian limit when $\ell \rightarrow \infty$.)

It follows from (22) that positions variables x^μ have a nontrivial Poisson bracket. To see this most easily, notice that with the help of the transformation

$$x^\mu \rightarrow \bar{x}^\mu, \quad \bar{x}^0 = x^0 - \ell p_i x^i, \quad \bar{x}^i = x^i, \quad (23)$$

one can diagonalize the kinetic Lagrangian (22), $\bar{L}^{kin} = \bar{x}^\mu \dot{p}_\mu$, so that the Poisson brackets in these new variables read

$$\{\bar{x}^\mu, \bar{x}^\nu\} = 0, \quad \{\bar{x}^\mu, p_\nu\} = \delta_\nu^\mu .$$

Using (23) one easily finds that

$$\{x^0, x^i\} = -\ell x^i, \quad \{x^i, x^j\} = 0, \quad (24)$$

$$\{x^0, p_0\} = 1, \quad \{x^i, p_j\} = \delta_j^i, \quad \{x^0, p_j\} = \ell p_j . \quad (25)$$

B. κ -momentum space, the κ -Poincaré Hopf algebra and κ -Minkowski spacetime

The characterization of the κ -momentum space given in the previous subsection is ideally suited for the purposes of our relative-locality studies, but it leaves partly implicit the connection with the κ -Poincaré Hopf algebra and its most popular applications in the study of spacetime noncommutativity. In order to expose more clearly this connection we shall now rederive the characterization of κ -momentum space already given in the previous subsection taking as starting point the role of the κ -Poincaré Hopf algebra in the study of the κ -Minkowski noncommutative spacetime[6, 21, 22].

Once again the selection of results we mention is due to the fact that the structure of the relative-locality framework of Refs. [1, 2], which we adopt, essentially requires some “inspiration” for an on-shell relation, a law of conservation of momentum at interactions, and some Poisson brackets. In this subsection we shall find this “inspiration” by revisiting the most studied formulation of theories in κ -Minkowski spacetime, often labeled as “bicrossproduct basis” [6, 22] or “time-to-the-right basis” [22, 24].

Starting from κ -Minkowski noncommutativity [6, 22], which can be thought of as the quantization of the Poisson brackets (25)

$$[\hat{x}^j, \hat{x}^0] = i\ell \hat{x}^j, \quad [\hat{x}^j, \hat{x}^k] = 0, \quad (26)$$

in this formulation one introduces the Fourier transform $\tilde{\Phi}(k)$ of a given κ -Minkowski field $\Phi(x)$ using the time-to-the-right convention

$$\Phi(\hat{x}) = \int d^4x \tilde{\Phi}(k) e^{ik_j \hat{x}^j} e^{ik_0 \hat{x}^0} .$$

This is often equivalently described in terms of the time-to-the-right Weyl map \mathcal{W}_R by writing that

$$\Phi(\hat{x}) = \mathcal{W}_R \left(\int d^4x \tilde{\Phi}(k) e^{ik_\mu x^\mu} \right),$$

where it is intended that coordinates trivially commute when placed inside the Weyl map, $\mathcal{W}_R(x^j x^0) = \mathcal{W}_R(x^0 x^j)$, and that taking a function out of the time-to-the-right Weyl map implies [24] time-to-the-right ordering, so that for example $\hat{x}^j \hat{x}^0 = \mathcal{W}_R(x^j x^0)$, $\hat{x}^j \hat{x}^0 = \mathcal{W}_R(x^0 x^j)$, and $e^{ik_j \hat{x}^j} e^{ik_0 \hat{x}^0} = \mathcal{W}_R(e^{ik_\mu x^\mu})$.

Then several arguments [6, 22], including the ones based on the recently-developed techniques of Noether analysis [25–27], lead one to find generators of symmetries under translations, space-rotations and boosts. For translations one has that

$$P_\mu e^{ik_j \hat{x}^j} e^{ik_0 \hat{x}^0} = k_\mu e^{ik_j \hat{x}^j} e^{ik_0 \hat{x}^0} = \mathcal{W}_R \left(-i \partial_\mu e^{ik_\nu x^\nu} \right) \quad (27)$$

and in general $P_\mu \mathcal{W}_R(f(x)) = \mathcal{W}_R(-i\partial_\mu f(x))$.

Similarly one has that the generators of space rotations are given by

$$R_l e^{ik_j \hat{x}^j} e^{ik_0 \hat{x}^0} = \varepsilon_{lmn} \hat{x}^m k_n e^{ik_j \hat{x}^j} e^{ik_0 \hat{x}^0}$$

and the generators of boosts are given by

$$\mathcal{N}_\ell e^{ik_j \hat{x}^j} e^{ik_0 \hat{x}^0} = \left[-\hat{x}^0 k_l + \hat{x}^l \left(\frac{1 - e^{2\ell k_0}}{2\ell} + \frac{\ell}{2} k_m k_m \right) \right] e^{ik_j \hat{x}^j} e^{ik_0 \hat{x}^0}.$$

These translations, rotations and boosts are found to be generators of the κ -Poincaré Hopf algebra, and their main properties are described, *e.g.*, in Refs. [6, 22, 24]. In particular one finds a deformed mass Casimir C_l , obtained from the generators given above

$$C_l = \left(\frac{2}{\ell} \right)^2 \sinh^2 \left(\frac{\ell}{2} P_0 \right) - e^{-\ell P_0} P_j P_j,$$

which can inspire a deformed on-shell relation for relativistic particles.

We also note the following properties of the translation generators

$$\begin{aligned} [P_l, \hat{x}^m] e^{ik_j \hat{x}^j} e^{ik_0 \hat{x}^0} &= -i \delta_l^m e^{ik_j \hat{x}^j} e^{ik_0 \hat{x}^0}, \\ [P_0, \hat{x}^0] e^{ik_j \hat{x}^j} e^{ik_0 \hat{x}^0} &= -i e^{ik_j \hat{x}^j} e^{ik_0 \hat{x}^0}, \\ [P_0, \hat{x}^l] e^{ik_j \hat{x}^j} e^{ik_0 \hat{x}^0} &= 0, \\ [P_l, \hat{x}^0] e^{ik_j \hat{x}^j} e^{ik_0 \hat{x}^0} &= -i \ell k_l e^{ik_j \hat{x}^j} e^{ik_0 \hat{x}^0} \end{aligned}$$

which should be compared to (25) of the previous subsection.

And the composition law which we derived in the previous subsection from the multiplication law on the group AN(3), is viewed in the spacetime-noncommutativity literature as a property of products of “time-to-the-right plane waves”,

$$e^{ik_j \hat{x}^j} e^{ik_0 \hat{x}^0} e^{ip_j \hat{x}^j} e^{ip_0 \hat{x}^0} = e^{ik_j \hat{x}^j} e^{i\ell k_0 p_j \hat{x}^j} e^{ik_0 \hat{x}^0} e^{ip_0 \hat{x}^0} = e^{i(k_j + e^{\ell k_0} p_j) \hat{x}^j} e^{i(k_0 + p_0) \hat{x}^0},$$

which indeed leads us once again to

$$(k \oplus p)_0 = k_0 + p_0,$$

$$(k \oplus p)_j = k_j + e^{\ell k_0} p_j.$$

Since this composition law can be derived within the time-to-the-right formulation of the κ -Poincaré/ κ -Minkowski framework, which first appeared in Ref. [22] by Majid and Ruegg, we shall refer to this composition law as the “Majid-Ruegg composition law” and to the associated affine connection on momentum space as the “Majid-Ruegg connection”.

IV. THE EXAMPLE OF κ -POINCARÉ-INSPIRED MOMENTUM SPACE WITH MAJID-RUEGG CONNECTION

Ref. [1] introduced the idea of using the geodesic distance from the origin to a generic point p_μ in momentum space \mathcal{P} as the mass of a particle. We shall here argue that according to this proposal one should view the κ -Poincaré/ κ -Minkowski framework as a case in which the metric on momentum space is de-Sitter like,

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -e^{-2\ell p_0} & 0 & 0 \\ 0 & 0 & -e^{-2\ell p_0} & 0 \\ 0 & 0 & 0 & -e^{-2\ell p_0} \end{pmatrix}, \quad (28)$$

and, as already anticipated, parallel transport is given in terms of the Majid-Ruegg connection.

The other objective of this section is to establish the torsion and nonmetricity of this κ -Poincaré-inspired setup.

A. Distance from the origin in a de Sitter momentum space

In order to calculate the geodesic distance from the origin to a generic point $p_\mu = (p_0, p_j)$ in momentum space \mathcal{P} we must find

$$D(0, p_\mu) = \int_0^1 ds \sqrt{g^{\mu\nu} \dot{p}_\mu \dot{p}_\nu} , \quad (29)$$

where p_μ is the solution of the geodesic equation

$$\ddot{p}_\rho + C_\rho^{\mu\nu} \dot{p}_\mu \dot{p}_\nu = 0 , \quad (30)$$

$g_{\mu\nu}$ is the metric of \mathcal{P} and $C_\rho^{\mu\nu}$ are the Christoffel symbols for the metric $g_{\mu\nu}$.

To find an approximate solution consider the metric slightly away from zero, which has the form

$$g^{\mu\nu} = \eta^{\mu\nu} + \ell h^{\mu\nu\rho} p_\rho + \dots \quad (31)$$

A simple calculation of the Christoffel symbols to the leading order,

$$C_\rho^{\mu\nu} = \frac{1}{2} g_{\rho\sigma} (g^{\sigma\mu,\nu} + g^{\nu\sigma,\mu} - g^{\mu\nu,\sigma}) = \frac{\ell}{2} \eta_{\rho\sigma} (h^{\sigma\mu\nu} + h^{\nu\sigma\mu} - h^{\mu\nu\sigma}) , \quad (32)$$

shows that the only non vanishing components are:

$$\begin{aligned} C_0^{ij} &= -\ell e^{-2\ell p_0} \delta_{ij} , \\ C_i^{j0} &= C_i^{0j} = -\ell \delta_{ij} , \end{aligned} \quad (33)$$

so that the geodesic equation (30) can be easily solved perturbatively with the boundary conditions

$$p_\mu(0) = 0, \quad p_\mu(1) = P_\mu . \quad (34)$$

The solution at leading order is

$$p_\rho(s) = P_\rho s + \frac{1}{2} C_\rho^{\mu\nu} P_\mu P_\nu (s - s^2) \quad (35)$$

and

$$\dot{p}_\rho(s) = P_\rho + \frac{1}{2} C_\rho^{\mu\nu} P_\mu P_\nu (1 - 2s) . \quad (36)$$

To compute the distance one must find

$$\sqrt{g^{\mu\nu} \dot{p}_\mu(s) \dot{p}_\nu(s)} = \sqrt{\eta^{\mu\nu} P_\mu P_\nu + C^{\rho\mu\nu} P_\rho P_\mu P_\nu (1 - 2s) + \ell h^{\mu\nu\rho} P_\rho P_\mu P_\nu s} . \quad (37)$$

To do that we use the identity that results from eq. (32)

$$C^{\rho\mu\nu} P_\rho P_\mu P_\nu = \frac{\ell}{2} h^{\rho\mu\nu} P_\rho P_\mu P_\nu . \quad (38)$$

So that finally we find

$$\sqrt{g^{\mu\nu} \dot{p}_\mu(s) \dot{p}_\nu(s)} = \sqrt{P^2 + C^{\rho\mu\nu} P_\rho P_\mu P_\nu} . \quad (39)$$

Integrating this from 0 to 1 and taking the square we get the final result

$$D(0, P_\mu) = m^2 = P^2 + C^{\rho\mu\nu} P_\rho P_\mu P_\nu . \quad (40)$$

Substituting the values of the connections found in eq. (33) we have

$$m^2 = P_0^2 - P_i^2 + \ell P_0 P_i^2 , \quad (41)$$

consistently with the leading-order form of the κ -Poincaré inspired on-shell relation.

B. Momentum space with de Sitter metric and Majid-Ruegg connection: torsion and (non)metricity

To further investigate the geometrical properties of momentum space we take the Majid-Ruegg composition law:

$$(p \oplus q)_0 = p_0 + q_0 ,$$

$$(k \oplus p)_j = p_j + e^{\ell p_0} q_j .$$

Using the Majid-Ruegg composition law, we can define a parallel transport on the momentum space \mathcal{P} as

$$(p \oplus dq)_\mu = p_\mu + dq_\mu - \Gamma_\mu^{\alpha\beta} p_\alpha dq_\beta + \dots \quad (42)$$

In particular for the (leading order) composition law

$$(p \oplus q)_0 = p_0 + q_0 ,$$

$$(p \oplus q)_i = p_i + q_i + \ell p_0 q_i , \quad (43)$$

we find that the only non-vanishing components of the connection are:

$$\Gamma_i^{0j} = -\ell \delta_i^j . \quad (44)$$

Given the components of the connection we can easily find the other geometric properties as torsion, nonmetricity and curvature.

For the torsion, we use [1]

$$T_\mu^{\alpha\beta} = -\frac{\partial}{\partial p_\alpha} \frac{\partial}{\partial q_\beta} (p \oplus q - q \oplus p)_\mu = 2\Gamma_\mu^{[\alpha\beta]} = \Gamma_\mu^{\alpha\beta} - \Gamma_\mu^{\beta\alpha} \quad (45)$$

to find that at leading order the only non vanishing components of the torsion tensor are

$$T_i^{0j} = -T_i^{j0} = \Gamma_i^{0j} = -\ell \delta_i^j . \quad (46)$$

And for the nonmetricity tensor,

$$N^{\alpha\mu\nu} = \nabla^\alpha g^{\mu\nu} = g^{\mu\nu, \alpha} + \Gamma_\beta^{\mu\alpha} g^{\beta\nu} + \Gamma_\beta^{\nu\alpha} g^{\mu\beta} , \quad (47)$$

the only non-vanishing components to the leading order are

$$N^{0ij} = 2\ell \delta^{ij} ,$$

$$N^{i0j} = \ell \delta^{ij} ,$$

$$N^{ij0} = \ell \delta^{ij} . \quad (48)$$

For what concerns the curvature of the connection, determined by [1]

$$R_\mu^{\alpha\beta\gamma} = 2 \frac{\partial}{\partial p_{[\alpha}} \frac{\partial}{\partial q_{\beta]}} \frac{\partial}{\partial r_\gamma} \left((p \oplus q) \oplus r - p \oplus (q \oplus r) \right)_\mu , \quad (49)$$

it is evident that it vanishes by construction in any leading-order analysis (in a power series in ℓ the first contribution to the curvature of the connection is of order ℓ^2). It is worth noticing that in the case of the Majid-Ruegg connection this curvature vanish exactly (to all orders) as a result of the fact that the Majid-Ruegg composition law is associative.

V. PARTIAL ANATOMY OF DISTANT RELATIVE-LOCALITY OBSERVERS

A. A starting point for the description of distant relative-locality observers

Let us now return to the preliminary results on translation invariance reported in Ref. [1], which we already briefly summarized in the first section, but we shall now analyze in greater detail. In Ref. [1] translation invariance was explicitly checked only for the idealized case of the process we already showed in Figure 1, with 3 particles of energy-momenta k_μ , p_μ , q_μ all incoming into the interaction.

Let us note down again here the action $\mathcal{S}^{example}$ which, according to Ref. [1], could describe the process in Figure 1:

$$\mathcal{S}^{example} = \int_{-\infty}^{s_0} ds \left(x^\mu \dot{k}_\mu + y^\mu \dot{p}_\mu + z^\mu \dot{q}_\mu + \mathcal{N}_k C[k] + \mathcal{N}_p C[p] + \mathcal{N}_q C[q] \right) - \xi^\mu \mathcal{K}_\mu(s_0), \quad (50)$$

where $C[k] = D^2(k) - m^2$ is the distance of k_μ from the origin of momentum space, and the on-shell condition is $C[k] = 0$, while the deformed law of energy-momentum conservation has been enforced by first introducing a connection-induced composition of the momenta,

$$\mathcal{K}_\mu(s) \equiv [k(s) \oplus p(s) \oplus q(s)]_\mu,$$

and then adding to the action a boundary term (in this case, at the $s = s_0$ boundary) with this \mathcal{K}_μ . The lagrange multipliers enforcing $\mathcal{K}_\mu = 0$ are denoted by ξ^μ and play the role of “interaction coordinates” in the sense of Ref. [1]. This is a theory on momentum space in the sense that the “particle coordinates” x^μ, y^μ, z^μ are introduced as “conjugate momenta of the momenta”, and for the action $\mathcal{S}^{example}$ one evidently has that

$$\begin{aligned} \{x^\mu, k_\nu\} &= \delta^\mu_\nu, \\ \{y^\mu, p_\nu\} &= \delta^\mu_\nu, \\ \{z^\mu, q_\nu\} &= \delta^\mu_\nu. \end{aligned} \quad (51)$$

Following again Ref. [1] we vary the action $\mathcal{S}^{example}$ keeping the momenta fixed at $s = \pm\infty$ (so that, for the case we are here considering, one has that $\delta k_\mu|_{s=-\infty} = 0, \delta p_\mu|_{s=-\infty} = 0, \delta q_\mu|_{s=-\infty} = 0$) and we find the equations of motion

$$\begin{aligned} \dot{k}_\mu &= 0, \quad \dot{p}_\mu = 0, \quad \dot{q}_\mu = 0, \\ C[k] &= 0, \quad C[p] = 0, \quad C[q] = 0, \\ \mathcal{K}_\mu &= 0, \\ \dot{x}^\mu &= \mathcal{N}_k \frac{\delta C[k]}{\delta k_\mu}, \quad \dot{y}^\mu = \mathcal{N}_p \frac{\delta C[p]}{\delta p_\mu}, \quad \dot{z}^\mu = \mathcal{N}_q \frac{\delta C[q]}{\delta q_\mu}, \end{aligned}$$

and the boundary conditions at the endpoints of the 3 semi-infinite worldlines

$$x^\mu(s_0) = \xi^\nu \frac{\delta \mathcal{K}_\nu}{\delta k_\mu}, \quad y^\mu(s_0) = \xi^\nu \frac{\delta \mathcal{K}_\nu}{\delta p_\mu}, \quad z^\mu(s_0) = \xi^\nu \frac{\delta \mathcal{K}_\nu}{\delta q_\mu}. \quad (52)$$

The relative locality is codified in the fact that for configurations with $\xi^\mu = 0$ the endpoints of the worldlines must coincide and be located in the origin of the observer ($x^\mu(s_0) = y^\mu(s_0) = z^\mu(s_0) = 0$), but for configurations such that $\xi^\mu \neq 0$ the endpoints of the worldlines do not coincide, since in general

$$\frac{\delta \mathcal{K}_\nu}{\delta k_\mu} \neq \frac{\delta \mathcal{K}_\nu}{\delta p_\mu} \neq \frac{\delta \mathcal{K}_\nu}{\delta q_\mu}, \quad (53)$$

so that in the coordinatization of the (in that case, distant) observer the interaction appears to be nonlocal.

As noticed in Ref. [1], taking as starting point of the analysis some observer Alice for whom³ $\xi_A^\mu \neq 0$, *i.e.* an observer distant from the interaction who sees the interaction as nonlocal, one can obtain from Alice an observer Bob for whom $\xi_B^\mu = 0$ if the transformation from Alice to Bob for endpoints of coordinates has the form

$$\begin{aligned} x_B^\mu(s_0) &= x_A^\mu(s_0) - \xi_A^\nu \frac{\delta \mathcal{K}_\nu(s)}{\delta k_\mu(s)} \Big|_{s=s_0}, \\ y_B^\mu(s_0) &= y_A^\mu(s_0) - \xi_A^\nu \frac{\delta \mathcal{K}_\nu(s)}{\delta p_\mu(s)} \Big|_{s=s_0}, \\ z_B^\mu(s_0) &= z_A^\mu(s_0) - \xi_A^\nu \frac{\delta \mathcal{K}_\nu(s)}{\delta q_\mu(s)} \Big|_{s=s_0}. \end{aligned} \quad (54)$$

³ When we compare two observers, Alice and Bob, we shall consistently use indices A and B to distinguish between quantities determined from one or the other. In particular, here we denote with ξ_A^μ the conservation-law Lagrange multipliers of observer Alice and with ξ_B^μ the corresponding Lagrange multipliers of observer Bob.

Such a property for the endpoint is produced of course, for the choice $b^\nu = \xi_A^\nu$, by the following corresponding prescription for the translation transformations:

$$\begin{aligned} x_B^\mu(s) &= x_A^\mu(s) - b^\nu \frac{\delta \mathcal{K}_\nu(s)}{\delta k_\mu(s)}, \\ y_B^\mu(s) &= y_A^\mu(s) - b^\nu \frac{\delta \mathcal{K}_\nu(s)}{\delta p_\mu(s)}, \\ z_B^\mu(s) &= z_A^\mu(s) - b^\nu \frac{\delta \mathcal{K}_\nu(s)}{\delta q_\mu(s)}, \\ \xi_B^\mu &= \xi_A^\mu - b^\mu. \end{aligned} \quad (55)$$

Indeed one finds by direct substitution that these transformations leave the equations of motion and the boundary conditions unchanged. And also the action is invariant; indeed

$$\begin{aligned} S_B^{example} &= \int_{-\infty}^{s_0} ds \left(x_B^\mu \dot{k}_\mu + y_B^\mu \dot{p}_\mu + z_B^\mu \dot{q}_\mu + \mathcal{N}_k C[k] + \mathcal{N}_p C[p] + \mathcal{N}_q C[q] \right) - \xi_B^\mu \mathcal{K}_\mu(s_0) \\ &= \int_{-\infty}^{s_0} ds \left(\left(x_A^\mu - b^\nu \frac{\delta \mathcal{K}_\nu}{\delta k_\mu} \right) \dot{k}_\mu + \left(y_A^\mu - b^\nu \frac{\delta \mathcal{K}_\nu}{\delta p_\mu} \right) \dot{p}_\mu + \left(z_A^\mu - b^\nu \frac{\delta \mathcal{K}_\nu}{\delta q_\mu} \right) \dot{q}_\mu + \mathcal{N}_k C[k] + \mathcal{N}_p C[p] + \mathcal{N}_q C[q] \right) - \xi_B^\mu \mathcal{K}_\mu(s_0) \\ &= S_A^{example, bulk} - \int_{-\infty}^{s_0} ds \frac{d}{ds} (b^\nu \mathcal{K}_\nu) - \xi_B^\mu \mathcal{K}_\mu(s_0) \\ &= S_A^{example, bulk} - (\xi_B^\mu + b^\mu) \mathcal{K}_\mu(s_0) \\ &= S_A^{example, bulk} - \xi_A^\mu \mathcal{K}_\mu(s_0) \\ &= S_A^{example}, \end{aligned} \quad (56)$$

where $S_A^{example, bulk}$ coincides with $S_A^{example}$ with the exception of boundary terms.

This also shows that all interactions are local according to nearby observers (observers themselves local to the interaction): if $\xi_A^\mu \neq 0$ for observer Alice, so that in Alice's coordinates the interaction is distant and nonlocal, one easily finds an observer Bob for whom $\xi_B^\mu = 0$, an observer local to the interaction who witnesses the interaction as a sharply local interaction in its origin.

For the purposes of the proposal we shall put forward in the following sections, it is important to notice here that these observations reported in Ref. [1] actually can be viewed as a prescription for translations generated by the “total momentum” \mathcal{K}_μ (in which however individual momenta are summed with a nonlinear composition law). In fact, in light of (51) the description of translation transformations given in (55) simply gives

$$\begin{aligned} \delta x_B^\mu(s) &= x_B^\mu(s) - x_A^\mu(s) = b^\nu \{ (k \oplus p \oplus q)_\nu, x^\mu \} = b^\nu \{ \mathcal{K}_\nu, x^\mu \} = -b^\nu \frac{\delta \mathcal{K}_\nu(s)}{\delta k_\mu(s)}, \\ \delta y_B^\mu(s) &= y_B^\mu(s) - y_A^\mu(s) = b^\nu \{ (k \oplus p \oplus q)_\nu, y^\mu \} = b^\nu \{ \mathcal{K}_\nu, y^\mu \} = -b^\nu \frac{\delta \mathcal{K}_\nu(s)}{\delta p_\mu(s)}, \\ \delta z_B^\mu(s) &= z_B^\mu(s) - z_A^\mu(s) = b^\nu \{ (k \oplus p \oplus q)_\nu, z^\mu \} = b^\nu \{ \mathcal{K}_\nu, z^\mu \} = -b^\nu \frac{\delta \mathcal{K}_\nu(s)}{\delta q_\mu(s)}. \end{aligned} \quad (57)$$

B. Some properties of our conservation laws

Our next task is to focus on another issue which also needs to be fully appreciated in order to work with relative locality: the issue of ordering momenta in the nonlinear composition law.

That ordering might be an issue is evident from the fact that relative-locality momentum spaces can in general allow [1] for interactions characterized by conservation laws which are possibly noncommutative (torsion) and/or non-associative (curvature of the connection). For leading-order analyses, of the type we are here motivating, only noncommutativity is possible, but that is enough to introduce quite some novelty with respect to standard absolute-locality theories. It should be noticed however that the number of truly different conservation laws is much smaller than one might naively imagine, as we shall now show for our illustrative κ -Poincaré-inspired example (a generalization of the argument shall be provided elsewhere [28]).

Let us first notice that while for arbitrary choices of k and p our composition law is evidently such that $k \oplus p \neq p \oplus k$ (noncommutativity), in the cases of interest when discussing interactions, cases in which the composition of momenta is used to

write a conservation law, we actually do have

$$k \oplus p = 0 \iff p \oplus k = 0 .$$

This is easily checked in the case which is of primary interest for us here:

$$0 = k_1 + p_1 + \ell k_0 p_1 = k_1 + p_1 + \ell p_0 k_1 ,$$

where on the right-hand-side we used in the leading-order correction the properties $k_0 = -p_0$ and $k_1 = -p_1$ which follow (at zero-th order) from $k \oplus p = 0$.

And actually $k \oplus p = 0 \iff p \oplus k = 0$ holds for any choice [28] of affine connection on momentum space, as shown by the following chain of properties:

$$k \oplus p = 0 \implies p = \ominus k \implies p \oplus k = \ominus k \oplus k = 0 .$$

This observation also simplifies the description of 3-particle interactions. In fact, since we have established that $k \oplus p = 0 \iff p \oplus k = 0$ it then evidently follows that⁴

$$k \oplus p \oplus q = 0 \iff q \oplus k \oplus p = 0 .$$

So, while there is no cyclicity property of the rule of composition of generic momenta, when the rule of composition is used for a conservation law it produces a conservation law with cyclicity.

C. Boundary terms and conservation of momenta

So we have seen that the number of truly independent conservation laws that can be postulated using the deformed composition law “ \oplus ” is smaller than one might have naively imagined, because of cyclicity. For some of the observations we report later on in this manuscript it is however important to appreciate that different compositions of momenta that (when set to zero) would produce the same conservation law still can lead to tangibly different choices of boundary terms enforcing the conservation laws.

Let us first illustrate the issue within the specific example of an interaction with two incoming and one outgoing particle, with conservation law

$$p \oplus k \oplus (\ominus q) = 0 .$$

This conservation law can be enforced by adding to the action a term of the form $\xi^\mu \mathcal{K}_\mu$, with $\mathcal{K}_\mu = [p \oplus k \oplus (\ominus q)]_\mu$ and ξ^μ are Lagrange multipliers. But this evidently is not the only choice of constraint term that enforces the chosen conservation law. For example let us observe that⁵

$$p \oplus k \oplus (\ominus q) = 0 \iff p \oplus k = q \iff (p \oplus k) - q = 0 ,$$

and also that

$$p \oplus k \oplus (\ominus q) = 0 \iff p \oplus k = q \iff \ominus q \oplus p \oplus k = \ominus q \oplus q = 0 .$$

So we see that the same conservation law⁶ can be enforced by adding a boundary term of the form $\xi^\mu \mathcal{K}_\mu$ with \mathcal{K}_μ given by any choice among $\mathcal{K}_\mu = [p \oplus k \oplus (\ominus q)]_\mu$, $\mathcal{K}_\mu = [(p \oplus k) - q]_\mu$, and $\mathcal{K}_\mu = [(\ominus q) \oplus p \oplus k]_\mu$. However, it is easy to verify (and this will play a role in the analysis reported in the following section) that these different possible choices of boundary terms enforcing the same momentum-conservation law actually produce boundary conditions that are physically different.

In the case of our interest, which is the case of the Majid-Ruegg connection, we shall be confronted with the observation that

$$(p \oplus k \oplus (\ominus q))_0 = ((p \oplus k) - q)_0 = ((\ominus q) \oplus p \oplus k)_0$$

but

$$((p \oplus k) - q)_1 = (p \oplus k \oplus (\ominus q))_1 + \ell(q_0 - k_0 - q_0)q_1 = ((\ominus q) \oplus p \oplus k)_1 + \ell q_0(q_1 - k_1 - q_1) .$$

However, when $((p \oplus k) - q)_\mu = 0$ one evidently also has⁷. (neglecting $O(\ell^2)$) that $\ell(q_0 - k_0 - q_0)q_1 = 0 = \ell q_0(q_1 - k_1 - q_1)$, so also for the specific case of the Majid-Ruegg connection one has this possibility of different boundary terms enforcing the same conservations laws, but producing physically-different boundary conditions.

⁴ Note that from $k \oplus p = 0 \iff p \oplus k = 0$, which holds for any choice of momentum-space affine connection (and associated composition law), it evidently follows that $(k \oplus p) \oplus q = 0 \iff q \oplus (k \oplus p) = 0$ but unless the composition law is associative this will not amount to a cyclicity property [28]. When, as in the case which is here of our primary interest, the composition law is associative we then have $q \oplus (k \oplus p) = (q \oplus k) \oplus p = q \oplus k \oplus p$ and a genuine cyclicity property arises.

⁵ This elementary chain of equivalence may at first appear striking since evidently in general $p \oplus k \oplus (\ominus q) \neq (p \oplus k) - q$, or, more precisely, if $p \oplus k \oplus (\ominus q) \neq 0$ then $p \oplus k \oplus (\ominus q) \neq (p \oplus k) - q$. However the chain of equivalences immediately follows upon observing that in the special cases where $p \oplus k \oplus (\ominus q) = 0$ (conservation laws) one then has that also $(p \oplus k) - q = 0$.

⁶ It should be noticed that in Ref. [7], where nonlinear conservation laws were analyzed from the perspective of the doubly-special-relativity research program, a possible role of such conservation laws of the type $p^{[in]} - (p^{[out,1]} \oplus p^{[out,2]}) = 0$ or $(p^{[in,1]} \oplus p^{[in,2]}) - (p^{[out,1]} \oplus p^{[out,2]}) = 0$ was already motivated on different grounds.

⁷ We stress again that the 3 conservation laws in question are exactly equivalent, equivalent to all orders in ℓ . We are however working here to leading order in ℓ , and for example the antipode \ominus for the Majid-Ruegg connection was here determined only to leading order. So the equivalence of the 3 conservation laws in question is of course verified within our computations only upon dropping subleading, $O(\ell^2)$, contributions.

D. A challenge for spacetime-translation invariance in theories on a relative-locality momentum space

We shall now characterize preliminarily the nature of some consistency conditions that should be enforced in order to produce a relativistic formulation with relative locality for interacting particles. As emphasized at the beginning of this section, in the relative-locality frameworks here of interest essentially what happens is that the correct notion of translation to distant observers must act on the endpoints of worldlines in a way that reflects the form of the boundary terms used to implement the conservation laws. As also shown at the beginning of this section this notion is never problematic for semi-infinite worldlines, with a single endpoint. But we must now highlight a challenge which materializes in all instances where two interactions are causally connected, *i.e.* there is a particle “exchanged” between the interactions, described by a finite worldline with two endpoints. In those instances we are going to have that the conservation laws essentially impose two conditions on the “exchanged worldline”, for the translation of the two endpoints. But we must request, for a relativistic description, that the worldline of distant observer Bob is solution of the same equations of motion that the initial observer Alice determines, and these relativistic demands are not automatically satisfied.

In order to render our concerns more explicit let us consider a specific example which does not admit the sort of relativistic description we are here interested in. For simplicity we consider a case in which the on-shell relation is undeformed and the symplectic structure is trivial. And we consider the situation shown in Figure 2, in which the two outgoing particles of a first decay themselves eventually decay.

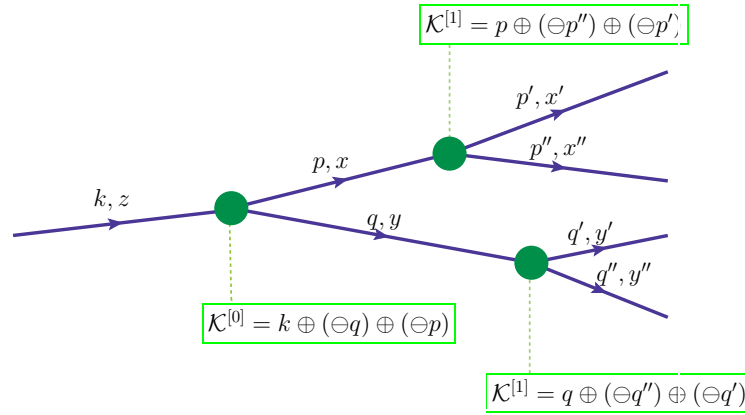


Figure 2. The action given and analyzed in this subsection would be intended for the description of the three causally-connected interactions shown here. But it appears that such a description is incompatible with a relativistic description of distant observers

A suitable description of the relevant conservation laws is the following:

$$\begin{aligned}
 0 &= (k \oplus (\ominus q) \oplus (\ominus p))_\mu = k_\mu - p_\mu - q_\mu - \ell \delta_\mu^j (k_0 q_j - q_0 q_j + k_0 p_j - p_0 p_j - q_0 p_j) , \\
 0 &= (p \oplus (\ominus p'') \oplus (\ominus p'))_\mu = p_\mu - p'_\mu - p''_\mu - \ell \delta_\mu^j (p_0 p''_j - p'_0 p''_j + p_0 p'_j - p'_0 p'_j - p'_0 p'_j) , \\
 0 &= (q \oplus (\ominus q'') \oplus (\ominus q'))_\mu = q_\mu - q'_\mu - q''_\mu - \ell \delta_\mu^j (q_0 q''_j - q'_0 q''_j + q_0 q'_j - q'_0 q'_j - q'_0 q'_j) .
 \end{aligned} \tag{58}$$

where for definiteness (it is easy to check that none of the points made in this subsection depend crucially on this choice) we specified as composition law the one coming from the “Majid-Ruegg connection”. Evidently the conservation laws concern a first interaction where a particle of momentum k decays into a particle of momentum p plus some other particle of momentum q , followed by two more decays, one where the particle of momentum p decays into particles of momentum p' and p'' and one where the particle of momentum q decays into particles of momentum q' and q'' .

The main observation we here want to convey is that the following choice of \mathcal{K} 's to be used in writing up constraints implementing the conservation laws

$$\begin{aligned}
 \mathcal{K}^{[0]}(s_0) &= k \oplus (\ominus q) \oplus (\ominus p) , \\
 \mathcal{K}^{[1]}(s_1) &= p \oplus (\ominus p'') \oplus (\ominus p') , \\
 \mathcal{K}^{[2]}(s_2) &= q \oplus (\ominus q'') \oplus (\ominus q') .
 \end{aligned}$$

which appears to be a very natural way to implement the conservation laws as constraints, does not lead to a relativistic description of distant observers.

To see this let us first write the action which would implement all this:

$$\begin{aligned}
S_A = & \int_{-\infty}^0 ds (z^\mu \dot{k}_\mu + \mathcal{N}_k [k^2 - m_a^2]) + \int_{s_0}^{s_1} ds (x^\mu \dot{p}_\mu + \mathcal{N}_p [p^2 - m_b^2]) \\
& + \int_{s_0}^{s_2} ds (y^\mu \dot{q}_\mu + \mathcal{N}_q [q^2 - m_c^2]) + \int_{s_1}^{+\infty} ds (x'^\mu \dot{p}'_\mu + \mathcal{N}_{p'} [p'^2 - m_d^2]) + \int_{s_1}^{+\infty} ds (x''^\mu \dot{p}''_\mu + \mathcal{N}_{p''} [p''^2 - m_e^2]) \\
& + \int_{s_2}^{+\infty} ds (y'^\mu \dot{q}'_\mu + \mathcal{N}_{q'} [q'^2 - m_f^2]) + \int_{s_2}^{+\infty} ds (y''^\mu \dot{q}''_\mu + \mathcal{N}_{q''} [q''^2 - m_g^2]) \\
& - \xi_{[0]A}^\mu \mathcal{K}_\mu^{[0]}(s_0) - \xi_{[1]A}^\mu \mathcal{K}_\mu^{[1]}(s_1) - \xi_{[2]A}^\mu \mathcal{K}_\mu^{[2]}(s_2) ,
\end{aligned} \tag{59}$$

where we restricted our focus on the undeformed on-shell condition $k^2 - m^2 = 0$ and we allowed for the presence of particles of different mass. The equations of motion that follow from varying this action evidently are:

$$\begin{aligned}
\dot{k}_\mu = \dot{p}_\mu = \dot{q}_\mu = \dot{p}'_\mu = \dot{p}''_\mu = \dot{q}'_\mu = \dot{q}''_\mu = 0 , \\
k^2 - m_a^2 = p^2 - m_b^2 = q^2 - m_c^2 = p'^2 - m_d^2 = p''^2 - m_e^2 = q'^2 - m_f^2 = q''^2 - m_g^2 = 0 , \\
\mathcal{K}_\mu^{[0]} = 0 , \quad \mathcal{K}_\mu^{[1]} = 0 , \quad \mathcal{K}_\mu^{[2]} = 0 , \\
\dot{z}^\mu = 2\mathcal{N}_k \delta_0^\mu k_0 - 2\mathcal{N}_k \delta_1^\mu k_1 , \quad \dot{x}^\mu = 2\mathcal{N}_p \delta_0^\mu p_0 - 2\mathcal{N}_p \delta_1^\mu p_1 , \quad \dot{x}'^\mu = 2\mathcal{N}_{p'} \delta_0^\mu p'_0 - 2\mathcal{N}_{p'} \delta_1^\mu p'_1 , \quad \dot{x}''^\mu = 2\mathcal{N}_{p''} \delta_0^\mu p''_0 - 2\mathcal{N}_{p''} \delta_1^\mu p''_1 , \\
\dot{y}^\mu = 2\mathcal{N}_q \delta_0^\mu q_0 - 2\mathcal{N}_q \delta_1^\mu q_1 , \quad \dot{y}'^\mu = 2\mathcal{N}_{q'} \delta_0^\mu q'_0 - 2\mathcal{N}_{q'} \delta_1^\mu q'_1 , \quad \dot{y}''^\mu = 2\mathcal{N}_{q''} \delta_0^\mu q''_0 - 2\mathcal{N}_{q''} \delta_1^\mu q''_1 .
\end{aligned}$$

And for the boundary conditions at endpoints of worldlines one finds:

$$\begin{aligned}
z_A^\mu(s_0) &= \xi_{[0]A}^\nu \frac{\delta \mathcal{K}_\nu^{[0]}}{\delta k_\mu} = \xi_{[0]A}^\mu - \ell \delta_0^\mu \xi_{[0]A}^1 (q_1 + p_1) , \\
x_A^\mu(s_0) &= -\xi_{[0]A}^\nu \frac{\delta \mathcal{K}_\nu^{[0]}}{\delta p_\mu} = \xi_{[0]A}^\mu - \ell \delta_0^\mu \xi_{[0]A}^1 p_1 + \ell \delta_1^\mu \xi_{[0]A}^1 (k_0 - q_0 - p_0) , \quad x_A^\mu(s_1) = \xi_{[1]A}^\nu \frac{\delta \mathcal{K}_\nu^{[1]}}{\delta p_\mu} = \xi_{[1]A}^\mu - \ell \delta_0^\mu \xi_{[0]A}^1 (p'_1 + p''_1) , \\
y_A^\mu(s_0) &= -\xi_{[0]A}^\nu \frac{\delta \mathcal{K}_\nu^{[0]}}{\delta q_\mu} = \xi_{[0]A}^\mu + \ell \delta_0^\mu \xi_{[0]A}^1 (q_1 + p_1) + \ell \delta_1^\mu \xi_{[0]A}^1 (k_0 - q_0) , \quad y_A^\mu(s_2) = \xi_{[2]A}^\nu \frac{\delta \mathcal{K}_\nu^{[2]}}{\delta q_\mu} = \xi_{[2]A}^\mu - \ell \delta_0^\mu \xi_{[0]A}^1 (q'_1 + q''_1) , \\
x_A'^\mu(s_1) &= -\xi_{[1]A}^\nu \frac{\delta \mathcal{K}_\nu^{[1]}}{\delta p'_\mu} = \xi_{[1]A}^\mu - \ell \delta_0^\mu \xi_{[0]A}^1 p'_1 + \ell \delta_1^\mu \xi_{[0]A}^1 (p_0 - p'_0 - p''_0) , \\
x_A''^\mu(s_1) &= -\xi_{[1]A}^\nu \frac{\delta \mathcal{K}_\nu^{[1]}}{\delta p''_\mu} = \xi_{[1]A}^\mu - \ell \delta_0^\mu \xi_{[0]A}^1 (p'_1 + p''_1) + \ell \delta_1^\mu \xi_{[0]A}^1 (p_0 - p''_0) , \\
y_A'^\mu(s_2) &= -\xi_{[2]A}^\nu \frac{\delta \mathcal{K}_\nu^{[2]}}{\delta q'_\mu} = \xi_{[2]A}^\mu - \ell \delta_0^\mu \xi_{[0]A}^1 q'_1 + \ell \delta_1^\mu \xi_{[0]A}^1 (q_0 - q'_0 - q''_0) , \\
y_A''^\mu(s_2) &= -\xi_{[2]A}^\nu \frac{\delta \mathcal{K}_\nu^{[2]}}{\delta q''_\mu} = \xi_{[2]A}^\mu - \ell \delta_0^\mu \xi_{[0]A}^1 (q'_1 + q''_1) + \ell \delta_1^\mu \xi_{[0]A}^1 (q_0 - q''_0) .
\end{aligned}$$

From this we immediately see that the action S_A does not admit a relativistic description of distant observers (in relative rest), at least not in the sense intended in Ref. [1]. And, as announced, the troubles originate from the finite worldlines, with two endpoints. For example, according to the observation reported in Ref. [1] (and here summarized in Subsec. V A), one would like translation transformations such that the endpoints of the worldline of momentum p transform as follows:

$$\begin{aligned}
x_B^\mu(s_0) &= x_A^\mu(s_0) + b^\nu \frac{\delta \mathcal{K}_\nu^{[0]}}{\delta p_\mu} , \\
x_B^\mu(s_1) &= x_A^\mu(s_1) - b^\nu \frac{\delta \mathcal{K}_\nu^{[1]}}{\delta p_\mu} .
\end{aligned} \tag{60}$$

But we also must insist, if the transformation from Alice to Bob is to be relativistic, that the equations of motion written by Alice and Bob are the same, so that in particular also for Bob $\dot{x}_B^\mu = 2\mathcal{N}_p \delta_0^\mu p_0 - 2\mathcal{N}_p \delta_1^\mu p_1$. However, enforcing both $\dot{x}_A^\mu = 2\mathcal{N}_p \delta_0^\mu p_0 - 2\mathcal{N}_p \delta_1^\mu p_1$ for Alice and $\dot{x}_B^\mu = 2\mathcal{N}_p \delta_0^\mu p_0 - 2\mathcal{N}_p \delta_1^\mu p_1$ for Bob imposes on our translation transformations that they be

rigid translations of the endpoints, in the sense that for (60) one should have

$$\frac{\delta \mathcal{K}_V^{[0]}}{\delta p_\mu} = -\frac{\delta \mathcal{K}_V^{[1]}}{\delta p_\mu} . \quad (61)$$

And it is easy to see that this condition, while automatically verified at zero-th order is in general not satisfied at $O(\ell)$.

For example for the Majid-Ruegg connection one has that

$$\begin{aligned} \mathcal{K}_1^{[0]} &= [k \oplus (\ominus q) \oplus (\ominus p)]_1 = k_1 - q_1 - p_1 - \ell(k_0 q_1 - q_0 q_1 + k_0 p_1 - q_0 p_1 - p_0 p_1) , \\ \mathcal{K}_1^{[1]} &= [p \oplus (\ominus p'') \oplus (\ominus p')]_1 = p_1 - p_1'' - p_1' - \ell(-p_0'' p_1' - p_0'' p_1' - p_0' p_1' + p_0 p_1'' + p_0 p_1') , \end{aligned} \quad (62)$$

from which it follows that

$$\begin{aligned} \frac{\delta \mathcal{K}_1^{[0]}}{\delta p_0} &= \ell p_1 , & \frac{\delta \mathcal{K}_1^{[0]}}{\delta p_1} &= -1 - \ell(k_0 - q_0 - p_0) , \\ \frac{\delta \mathcal{K}_1^{[1]}}{\delta p_0} &= -\ell(p_1' + p_1'') , & \frac{\delta \mathcal{K}_1^{[1]}}{\delta p_1} &= 1 . \end{aligned} \quad (63)$$

which indeed confirms that the condition (61) is satisfied at zero-th order but violated at $O(\ell)$.

VI. KNOWN RELATIVE-LOCALITY RESULTS FOR FREE κ -POINCARÉ PARTICLES IN HAMILTONIAN DESCRIPTION

The insight gained in the previous section is going to guide us, in the next section, to a satisfactory relativistic description of interacting particles, with relative locality, applicable also to cases where particles are “exchanged”, *i.e.* there are finite worldlines. As a further element of preparation for that task we find it useful to briefly review the recent results on the relative locality produced by a “ κ -Poincaré inspired Hamiltonian” description of free particles. This is because part of our confidence in the way we shall propose to proceed for the Lagrangian description of interacting particles is provided by exposing a consistency with these pre-existing Hamiltonian free-particle results.

Also for this aside on Hamiltonian description of free particles on a “ κ -Minkowski phase space” we introduce an auxiliary worldline parameter s and we denote by \dot{Q} the s derivative of an observable Q , so that $\dot{Q} \equiv \partial Q / \partial s$.

On the basis of what was derived in the earlier Section III our “ κ -Minkowski phase-space ansatz” is such that the Poisson bracket for the spacetime coordinates is

$$\{x^1, x^0\} = \ell x^1, \quad (64)$$

spacetime translations are governed by

$$\{x^0, p_0\} = 1, \quad \{x^1, p_0\} = 0, \quad (65)$$

$$\{x^0, p_1\} = \ell p_1, \quad \{x^1, p_1\} = 1. \quad (66)$$

and the on-shell relation is

$$m^2 = p_0^2 - p_1^2 + \ell p_0 p_1^2. \quad (67)$$

One can then use [29, 30]

$$\mathcal{H}_p = \mathcal{N}_p C[p] = \mathcal{N}_p (p_0^2 - p_1^2 + \ell p_0 p_1^2 - m^2)$$

as Hamiltonian of evolution of the observables on the worldline of a particle in terms of the worldline parameter s .

Hamilton’s equations evidently give the conservation of p_0 and p_1 along the worldlines. And concerning worldlines one finds that

$$\begin{aligned} \dot{x}^0 &= \{x^0, \mathcal{H}_p\} = \frac{\partial \mathcal{H}_p}{\partial p_0} \{x^0, p_0\} + \frac{\partial \mathcal{H}_p}{\partial p_1} \{x^0, p_1\} = \mathcal{N}_p (2p_0 - \ell p_1^2) , \\ \dot{x}^1 &= \{x^1, \mathcal{H}_p\} = \frac{\partial \mathcal{H}_p}{\partial p_0} \{x^1, p_0\} + \frac{\partial \mathcal{H}_p}{\partial p_1} \{x^1, p_1\} = -2\mathcal{N}_p (p_1 - \ell p_0 p_1) , \end{aligned}$$

so that the velocity is⁸

$$v = \frac{\dot{x}^1}{\dot{x}^0} = -\frac{p_1}{p_0} \left(1 - \ell p_0 + \frac{1}{2} \ell \frac{p_1^2}{p_0} \right) = -\frac{p_1}{\sqrt{p_1^2 + m^2}} + \ell p_1 \frac{m^2}{p_1^2 + m^2}, \quad (68)$$

where, in light of Eq. (67),

$$p_0 = \sqrt{p_1^2 + m^2} - \frac{1}{2} \ell p_1^2. \quad (69)$$

The worldlines then are

$$x^1(p_1, \bar{x}^1, \bar{x}^0; x^0) = \bar{x}^1 - \left(\frac{p_1}{\sqrt{p_1^2 + m^2}} - \ell p_1 \frac{m^2}{p_1^2 + m^2} \right) (x^0 - \bar{x}^0).$$

In particular, for massless particles these worldlines give a momentum-independent particle speed:

$$x^1(p_1, \bar{x}^1, \bar{x}^0; x^0) = \bar{x}^1 - \frac{p_1}{|p_1|} (x^0 - \bar{x}^0).$$

However, as noticed in Ref. [18], the fact that worldlines of massless particles are characterized by “coordinate velocities” which are momentum independent does not ensure that simultaneously-emitted massless particles of different momentum are detected simultaneously. One must factor in the anomalous properties of translations in κ -Minkowski, and this is where the relativity of locality is most vividly exposed.

To see this it suffices to consider a simultaneous emission occurring in the origin of an observer Alice. This will be described by Alice in terms of two worldlines, a massless particle with momentum p_1^s and a massless particle with momentum p_1^h , which actually coincide because of the momentum independence of the coordinate velocity:

$$x_{[A]p^s}^1(x_{[A]}^0) = x_{[A]}^0, \quad x_{[A]p^h}^1(x_{[A]}^0) = x_{[A]}^0 \quad (70)$$

(where we took both $p_1^h < 0$ and $p_1^s < 0$, so that the particles propagate along the positive direction of the x^1 axis).

It is useful to focus on the case of p_1^s and p_1^h such that $|p_1^s| \ll |p_1^h|$, and $|\ell p_1^s| \simeq 0$ (the particle with momentum p_1^s is soft enough that it behaves as if $\ell = 0$) while $|\ell p_1^h| \neq 0$, in the sense that for the hard particle the effects of ℓ -deformation are not negligible.

Then we need to use the fact that the assignments of coordinates on points of a worldline adopted by two observers connected by a generic translation \mathcal{T}_{b^0, b^1} , with component b^0 along the x^0 axis and component b^1 along the x^1 axis, is such that

$$\begin{aligned} x'^1 &= x^1 + b^0 \{p_0, x^1\} + b^1 \{p_1, x^1\}, \\ x'^0 &= x^0 + b^0 \{p_0, x^0\} + b^1 \{p_1, x^0\}. \end{aligned}$$

Using these we can look [18] at the two Alice worldlines, given in (70), from the perspective of a second observer, Bob, at rest with respect to Alice at distance b from Alice (Bob = \mathcal{T}_{b, b^0} Alice), local to a detector that the two particles eventually reach. Of course, in light of the form of the worldlines, according to Alice’s coordinates the two particles reach Bob simultaneously. But can this distant coincidence of events be trusted? The two events which according to the coordinates of distant observer Alice are coincident are the crossing of Bob’s worldline with the worldline of the particle with momentum p_1^s and the crossing of Bob’s worldline with the worldline of the particle with momentum p_1^h . To clarify the situation we should look at the two worldlines from the perspective of Bob, the observer who is local to the detection of the particles.

Evidently these Bob worldlines are obtained from Alice worldlines using the translation transformation codified in (65), (66). Acting on a generic Alice worldline $x_{[A]}^1(p_1, \bar{x}_{[A]}^1, \bar{x}_{[A]}^0; x_{[A]}^0)$ this gives a Bob worldline $x_{[B]}^1(p_1, \bar{x}_{[B]}^1, \bar{x}_{[B]}^0; x_{[B]}^0)$ as follows:

$$\begin{aligned} x_{[B]}^1 &= x_{[A]}^1 + b \{p_0, x_{[A]}^1\} + b \{p_1, x_{[A]}^1\} = x_{[A]}^1 - b, \\ x_{[B]}^0 &= x_{[A]}^0 + b \{p_0, x_{[A]}^0\} + b \{p_1, x_{[A]}^0\} = x_{[A]}^0 - b - \ell b p_1. \end{aligned}$$

⁸ Note that with our choice of conventions (signature of the metric) a particle on shell moving along the positive direction of the x^1 axis has positive v^1 and negative p_1 .

And specifically for the two worldlines of our interest, given for Alice in (70), one then finds

$$\begin{aligned} x_{[B]p^s}^1(x_{[B]}^0) &= x_{[B]}^0 + \ell b p_1^s \simeq x_{[B]}^0, \\ x_{[B]p^h}^1(x_{[B]}^0) &= x_{[B]}^0 + \ell b p_1^h. \end{aligned}$$

The two worldlines, which were coincident according to Alice, are distinct worldlines for Bob. And it is established that [18] according to Bob, who is at the detector, the two particles reach the detector at different times: $x_{[B]}^0 \simeq 0$ for the soft particle and $x_{[B]}^0 = -\ell b p_1^h = \ell b |p_1^h|$ for the hard particle. This for the two massless particles which, according to the observer Alice who is at the emitter, were emitted simultaneously. The difference of times of detection at Bob is governed by the simple formula

$$\Delta t = \ell b |\Delta p_1|. \quad (71)$$

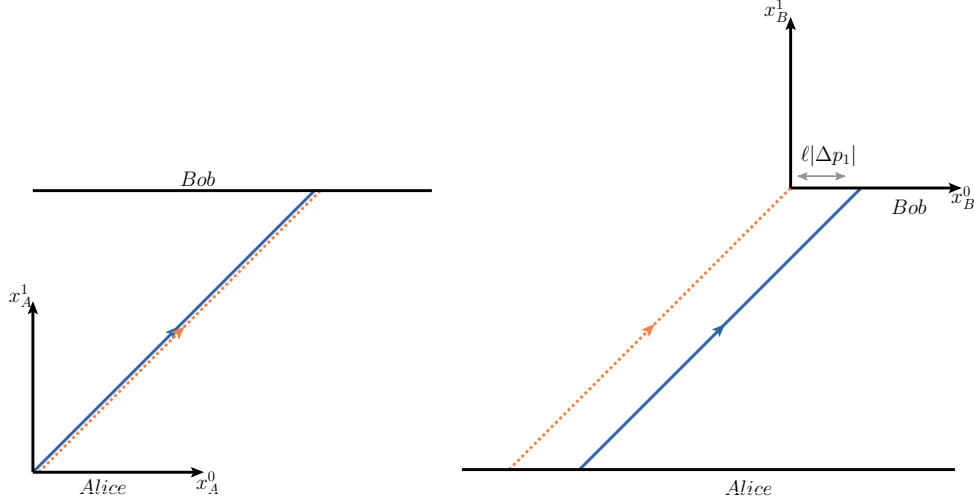


Figure 3. Two simultaneously-emitted massless particles of different momentum in κ -Minkowski are detected at different times. The figure shows how the simultaneous emission of two such particles and their non-simultaneous detection is described in the coordinates of observer Alice (left panel), who is at the emitter, and in the coordinates of observer Bob (right panel), who is at the detector.

VII. A LAGRANGIAN DESCRIPTION OF RELATIVE LOCALITY WITH INTERACTIONS

A. κ -Minkowski symplectic structure and translations generated by total momentum

In this section we show that it is possible to have a relativistic description of pairs of distant observers (in relative rest), in descriptions of interactions with particle exchanges (finite worldlines) formulated within the relative-locality framework of Refs. [1, 2]. The main challenge we shall face in this section is the one characterized in Subsection V D: relativistic descriptions of a single interaction with relative locality are rather elementary, but when pairs of interactions a causally connected the availability of a relativistic description for distant observers is in no way assured, and actually before the study we are here reporting there was no known example where it had been shown to work.

We shall also find reassuring that the Lagrangian description we obtain for interacting particles, reproduces in an appropriate limit the known results reviewed in the previous section, concerning relative locality in a κ -Poincaré-inspired Hamiltonian description of free particles. In doing so we also provide an explicit analysis in which the non-trivial geometry of momentum space is analyzed while adopting a non-standard symplectic structure.

Indeed, the first point of contact between our Lagrangian description and the Hamiltonian description reviewed in the previous section is found in the choice of symplectic structure and on-shell condition characterizing the “free part” of the action”, which for the case of 3 particles (of momenta k_μ incoming and momenta p_μ and q_μ outgoing) takes the form:

$$\begin{aligned} S_\kappa^{bulk} &= \int_{-\infty}^{s_0} ds (z^\mu \dot{k}_\mu - \ell z^1 k_1 \dot{k}_0 + \mathcal{N}_k C_\kappa[k]) + \int_{s_0}^{+\infty} ds (x^\mu \dot{p}_\mu - \ell x^1 p_1 \dot{p}_0 + \mathcal{N}_p C_\kappa[p]) \\ &\quad + \int_{s_0}^{+\infty} ds (y^\mu \dot{q}_\mu - \ell y^1 q_1 \dot{q}_0 + \mathcal{N}_q C_\kappa[q]), \end{aligned} \quad (72)$$

where

$$C_\kappa[k] \equiv k_0^2 - k_1^2 + \ell k_0 k_1^2 ,$$

so that we implement the on-shell relation of the Hamiltonian κ -Minkowski phase-space setup reviewed in the previous section. We are adopting κ -Minkowski Poisson brackets, so that for example for x^μ

$$\{x^1, x^0\} = \ell x^1 ,$$

and from (72) one recognizes that our symplectic structure also matches the one of the Hamiltonian κ -Minkowski phase-space setup reviewed in the previous section; so that for example for x^μ, p_μ

$$\begin{aligned} \{x^0, p_0\} &= 1, & \{x^1, p_0\} &= 0, \\ \{x^0, p_1\} &= \ell p_1, & \{x^1, p_1\} &= 1. \end{aligned}$$

For what concerns the conservation laws at interactions we shall adopt the Majid-Ruegg connection. But, as evident on the basis of the observation we reported in Section V, once the conservation laws are specified the construction of this type of relative-locality theory still leaves open a choice among possible alternative ways of implementing such laws of momentum conservation through some boundary terms. We adopt a particular choice which we favor because it happens to be immune from the problem here highlighted in Subsection V D, which instead is found to affect several alternative possibilities [31, 32]. We qualify our choice of momentum-conservation constraints as the ones that are suitable for a description of translations in which “translations are generated by the total momentum”, for reasons that will become clearer in the reminder of this section. The prescription we adopt will be generalized as we go along, but let us here start with the case of a single interaction, whose conservation law is

$$0 = k \oplus (\ominus q) \oplus (\ominus p) .$$

As already stressed in Subsection V C, such a conservation law could be implemented by several inequivalent choices of $\mathcal{K}^{[0]}$ for the constraints on the endpoints of worldlines, including

$$\mathcal{K}^{[0]} = k \oplus (\ominus q) \oplus (\ominus p) ,$$

$$\mathcal{K}^{[0]} = (\ominus q) \oplus (\ominus p) \oplus k ,$$

$$\mathcal{K}^{[0]} = k - (p \oplus q) .$$

We find that this latter option $\mathcal{K}^{[0]} = k - (p \oplus q)$ admits a consistent relativistic description of distant observers. Evidence of this will be provided throughout this section. But let us first notice that this sort of constraints is very intuitive: they implement the rather standard concept that the conservation law is such that the total momentum before an interaction should equal the total momentum after an interaction. And we shall show that this form of the constraints allows one to preserve the usual notion that translation transformations are generated by the total momentum (though of course in our case the total momentum is obtained in terms of the nonlinear composition law), even when several interactions are analyzed and particles are exchanged among some of the interactions.

Essentially our proposal establishes that there is at least one way (at present we are unable to claim that it is unique) to address the challenge we earlier highlighted in Eq. (61). And the conceptual content of the solution we found for addressing that challenge exemplified in Eq. (61) is, as shown below, rather simple: the most basic notion of relativistic translation transformation is as usual generated by the total momentum acting on worldlines, but (as also shown in our discussion surrounding Eq. (61)) the boundary terms used to enforce the conservation laws require that endpoints transform under translations in ways governed by (or at least conditioned by) the boundary terms. We handle the challenge illustrated by Eq. (61) by essentially finding a way to render these two demands compatible: we enforce the conservation laws through boundary terms written in such a way that when the worldlines are translated by the total momentum then the endpoints automatically match the demands of the boundary terms.

Let us start seeing how this plays out for a case with a single interaction, considering, for the interaction in Figure 4, the action

$$\begin{aligned} S^\kappa &= S_{bulk}^\kappa + S_{int}^\kappa = \int_{-\infty}^{s_0} ds (z^\mu \dot{k}_\mu - \ell z^1 k_1 \dot{k}_0 + \mathcal{N}_k C_\kappa[k]) + \int_{s_0}^{+\infty} ds (x^\mu \dot{p}_\mu - \ell x^1 p_1 \dot{p}_0 + \mathcal{N}_p C_\kappa[p]) \\ &\quad + \int_{s_0}^{+\infty} ds (y^\mu \dot{q}_\mu - \ell y^1 q_1 \dot{q}_0 + \mathcal{N}_q C_\kappa[q]) - \xi_{[0]}^\mu \mathcal{K}_\mu^{[0]}(s_0) , \end{aligned} \tag{73}$$

where indeed for $\mathcal{K}^{[0]}$ we take

$$\mathcal{K}_\mu^{[0]}(s_0) = k_\mu - (p \oplus q)_\mu = k_\mu - p_\mu + \ell \delta_\mu^1 q_0 p_1 . \quad (74)$$

The equations of motion that follow from our action S^κ are of course the same found in the Hamiltonian formulation of free κ -Minkowski particles reviewed in the previous subsection:

$$\begin{aligned} \dot{p}_\mu &= 0 , \quad \dot{q}_\mu = 0 , \quad \dot{k}_\mu = 0 , \\ C_\kappa[p] &= 0 , \quad C_\kappa[q] = 0 , \quad C_\kappa[k] = 0 , \\ \mathcal{K}_\mu^{[0]}(s_0) &= 0 \end{aligned} \quad (75)$$

$$\begin{aligned} \dot{x}^\mu &= \mathcal{N}_p \left(\frac{\delta C_\kappa[p]}{\delta p_\mu} + \ell \delta_0^\mu \frac{\delta C_\kappa[p]}{\delta p_1} p_1 \right) = \delta_0^\mu \mathcal{N}_p (2p_0 - \ell p_1^2) - 2\delta_1^\mu \mathcal{N}_p (p_1 - \ell p_0 p_1) , \\ \dot{y}^\mu &= \mathcal{N}_q \left(\frac{\delta C_\kappa[q]}{\delta q_\mu} + \ell \delta_0^\mu \frac{\delta C_\kappa[q]}{\delta q_1} q_1 \right) = \delta_0^\mu \mathcal{N}_q (2q_0 - \ell q_1^2) - 2\delta_1^\mu \mathcal{N}_q (q_1 - \ell q_0 q_1) , \\ \dot{z}^\mu &= \mathcal{N}_k \left(\frac{\delta C_\kappa[k]}{\delta k_\mu} + \ell \delta_0^\mu \frac{\delta C_\kappa[k]}{\delta k_1} k_1 \right) = \delta_0^\mu \mathcal{N}_k (2k_0 - \ell k_1^2) - 2\delta_1^\mu \mathcal{N}_k (k_1 - \ell k_0 k_1) . \end{aligned} \quad (76)$$

And the interaction at $s = s_0$ produces the boundary conditions:

$$\begin{aligned} x^\mu(s_0) &= -\xi_{[0]}^\nu \left(\frac{\delta \mathcal{K}_\nu^{[0]}}{\delta p_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_\nu^{[0]}}{\delta p_1} p_1 \right) = \xi_{[0]}^\mu + \ell \delta_0^\mu \xi_{[0]}^1 (p_1 + q_1) , \\ y^\mu(s_0) &= -\xi_{[0]}^\nu \left(\frac{\delta \mathcal{K}_\nu^{[0]}}{\delta q_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_\nu^{[0]}}{\delta q_1} q_1 \right) = \xi_{[0]}^\mu + \ell \delta_0^\mu \xi_{[0]}^1 q_1 + \ell \delta_0^\mu \xi_{[0]}^1 p_0 , \\ z^\mu(s_0) &= \xi_{[0]}^\nu \left(\frac{\delta \mathcal{K}_\nu^{[0]}}{\delta k_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_\nu^{[0]}}{\delta k_1} k_1 \right) = \xi_{[0]}^\mu + \ell \delta_0^\mu \xi_{[0]}^1 k_1 . \end{aligned} \quad (77)$$

The mechanism for relative locality which we already discussed above is evidently also present here: the boundary conditions establish that if the observer is local to the interaction, *i.e.* $\xi_{[0]}^\mu = 0$, then all endpoints of the semiinfinite worldlines are in the origin of the observer. If instead $\xi_{[0]}^\mu \neq 0$ the endpoints of worldlines do not coincide.

It is also easy to check that our equations of motion (75), (76) and boundary conditions (77) invariant under deformed trans-

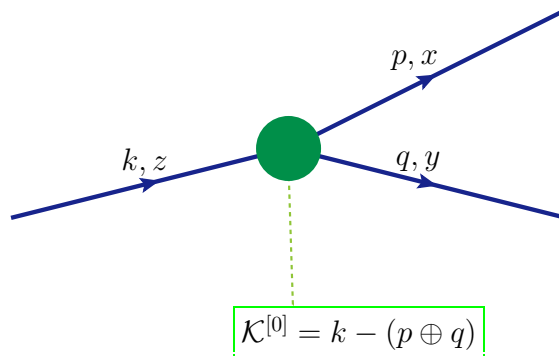


Figure 4. The choice of \mathcal{K} we adopt for the case of a single interaction with 1 incoming and 2 outgoing particles.

lations generated by the total momentum acting on coordinates

$$\begin{aligned}
x_B^0(s) &= x_A^0(s) + b^\mu \{ (p \oplus q)_\mu, x^0 \} = x_A^0(s) - b^0 - \ell b^1 (p_1 + q_1) , \\
x_B^1(s) &= x_A^1(s) + b^\mu \{ (p \oplus q)_\mu, x^1 \} = x_A^1(s) - b^1 , \\
y_B^0(s) &= y_A^0(s) + b^\mu \{ (p \oplus q)_\mu, y^0 \} = y_A^0(s) - b^0 - \ell b^1 q_1 , \\
y_B^1(s) &= y_A^1(s) + b^\mu \{ (p \oplus q)_\mu, y^1 \} = y_A^1(s) - b^1 - \ell b^1 p_0 , \\
z_B^0(s) &= z_A^0(s) + b^\mu \{ k_\mu, z^0 \} = z_A^0(s) - b^0 - \ell b^1 k_1 , \\
z_B^1(s) &= z_A^1(s) + b^\mu \{ k_\mu, z^1 \} = z_A^1(s) - b^1 .
\end{aligned} \tag{78}$$

It should be noticed that we essentially prescribe that a given point of a given worldline is translated by acting with the total momentum written in the way that is appropriate for that point of the worldline, so that, in the specific example here under consideration, all points with $s < s_0$ are translated by k_μ whereas all points with $s > s_0$ are translated by $(p \oplus q)_\mu$.

The invariance of the equations of motion is easily seen by observing that Eq. (75) guarantees that $\dot{p}_\mu = 0$, $\dot{q}_\mu = 0$, $\dot{k}_\mu = 0$ and that the translation transformations depend only on momenta. Considering for example the worldline x^μ , and assuming of course that both observer Alice and observer Bob adopt the equations of motion

$$\begin{aligned}
\dot{x}_{[A]}^\mu &= \mathcal{N}_p \left(\frac{\delta C_\kappa[p]}{\delta p_\mu} + \ell \delta_0^\mu \frac{\delta C_\kappa[p]}{\delta p_1} p_1 \right) , \\
\dot{x}_{[B]}^\mu &= \mathcal{N}_p \left(\frac{\delta C_\kappa[p]}{\delta p_\mu} + \ell \delta_0^\mu \frac{\delta C_\kappa[p]}{\delta p_1} p_1 \right) ,
\end{aligned} \tag{79}$$

one indeed finds that the translation transformations

$$\begin{aligned}
x_B^0(s) &= x_A^0(s) - b^0 - \ell b^1 (p_1 + q_1) , \\
x_B^1(s) &= x_A^1(s) - b^1
\end{aligned} \tag{80}$$

are such that $\dot{x}_{[B]}^\mu = \dot{x}_{[A]}^\mu$ (since momenta are conserved).

And the invariance of the boundary conditions is easily seen by directly checking that the boundary conditions for Alice are mapped by the translation transformations into the (identical) boundary conditions for Bob. For example, we have for Alice

$$\dot{x}_{[A]}^\mu(s_0) = -\xi_{[0]A}^\nu \left(\frac{\delta \mathcal{K}_\nu^{[0]}}{\delta p_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_\nu^{[0]}}{\delta p_1} p_1 \right) = \xi_{[0]A}^\mu + \ell \delta_0^\mu \xi_{[0]A}^1 (p_1 + q_1) , \tag{81}$$

and the translation transformations (80) map this into

$$\begin{aligned}
\dot{x}_{[B]}^\mu(s_0) &= \dot{x}_{[A]}^\mu(s_0) - b^\mu - \ell \delta_0^\mu (p_1 + q_1) = -\xi_{[0]A}^\nu \left(\frac{\delta \mathcal{K}_\nu^{[0]}}{\delta p_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_\nu^{[0]}}{\delta p_1} p_1 \right) - b^\mu - \ell \delta_0^\mu (p_1 + q_1) \\
&= \xi_{[0]A}^\mu + \ell \delta_0^\mu \xi_{[0]A}^1 (p_1 + q_1) - b^\mu - \ell \delta_0^\mu (p_1 + q_1) = -(\xi_{[0]A}^\mu - b^\mu) \left(\frac{\delta \mathcal{K}_\nu^{[0]}}{\delta p_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_\nu^{[0]}}{\delta p_1} p_1 \right) \\
&= -\xi_{[0]B}^\nu \left(\frac{\delta \mathcal{K}_\nu^{[0]}}{\delta p_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_\nu^{[0]}}{\delta p_1} p_1 \right) .
\end{aligned} \tag{82}$$

Besides checking the invariance of the equations of motion and the boundary conditions, which however already ensure that our translations are physical symmetries, it is also valuable to apply the translation transformations (78) to the action (73), so that we can find the relation between the action of Alice and the action of Bob (distant from Alice). We find

$$\begin{aligned}
S_B^\kappa &= S_A^\kappa + \int_{-\infty}^{s_0} ds (-b^\mu \dot{k}_\mu) + \int_{s_0}^{\infty} ds (-b^\mu \dot{p}_\mu - \ell b^1 q_1 \dot{p}_0) \\
&\quad + \int_{s_0}^{\infty} ds (-b^\mu \dot{q}_\mu - \ell b^1 p_0 \dot{q}_1) + \Delta \xi_{[0]}^\mu \mathcal{K}_\mu^{[0]}(s_0) .
\end{aligned}$$

where $\Delta \xi_{[0]}^\mu = \xi_{[0]B}^\mu - \xi_{[0]A}^\mu$. Substituting $s' = -s$ in the first integral and then relabeling $s' \rightarrow s$, one then gets

$$\Delta S^\kappa = S_B^\kappa - S_A^\kappa = \int_{s_0}^{\infty} ds (b^\mu (\dot{k}_\mu - \dot{p}_\mu - \dot{q}_\mu) - \ell b^1 (q_1 \dot{p}_0 + p_0 \dot{q}_1)) + \Delta \xi_{[0]}^\mu \mathcal{K}_\mu^{[0]}(s_0) .$$

Then using Eq. (74) we find

$$\Delta S^\kappa = S_B^\kappa - S_A^\kappa = \int_{s_0}^{\infty} ds \frac{d}{ds} \left[b^\mu \mathcal{K}_\mu^{[0]} \right] + \Delta \xi_{[0]}^\mu \mathcal{K}_\mu^{[0]}(s_0) . \quad (83)$$

The total derivatives contribute to the boundaries in such a way that, for the difference (83) to be null, it must hold

$$\left(\Delta \xi_{[0]}^\mu - b^\mu \right) \mathcal{K}_\mu^{[0]}(s_0) ,$$

from which we see that the $\xi_{[0]}^\mu$ translate classically:

$$\xi_{[0]B}^\mu = \xi_{[0]A}^\mu - b^\mu . \quad (84)$$

And when the observer Alice is distant from the interaction, *i.e.* $\xi_{[0]A}^\mu \neq 0$, one can always find through such translation transformations an observer Bob local to the interaction and for whom the endpoints of worldlines match:

$$\begin{aligned} x_B^\mu(s_0) &= -\xi_B^\nu \left(\frac{\delta \mathcal{K}_\nu^{[0]}}{\delta p_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_\nu^{[0]}}{\delta p_1} p_1 \right) = -(\xi_A^\nu - b^\nu) \left(\frac{\delta \mathcal{K}_\nu^{[0]}}{\delta p_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_\nu^{[0]}}{\delta p_1} p_1 \right) = 0 , \\ y_B^\mu(s_0) &= -\xi_B^\nu \left(\frac{\delta \mathcal{K}_\nu^{[0]}}{\delta q_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_\nu^{[0]}}{\delta q_1} q_1 \right) = -(\xi_A^\nu - b^\nu) \left(\frac{\delta \mathcal{K}_\nu^{[0]}}{\delta q_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_\nu^{[0]}}{\delta q_1} q_1 \right) = 0 , \\ z_B^\mu(s_0) &= \xi_B^\nu \left(\frac{\delta \mathcal{K}_\nu^{[0]}}{\delta k_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_\nu^{[0]}}{\delta k_1} k_1 \right) = (\xi_A^\nu - b^\nu) \left(\frac{\delta \mathcal{K}_\nu^{[0]}}{\delta k_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_\nu^{[0]}}{\delta k_1} k_1 \right) = 0 . \end{aligned} \quad (85)$$

B. Causally connected interactions and translations generated by total momentum

Our next challenge is to deal with causally-connected interactions. We show in figure the case we here analyze as illustrative example.

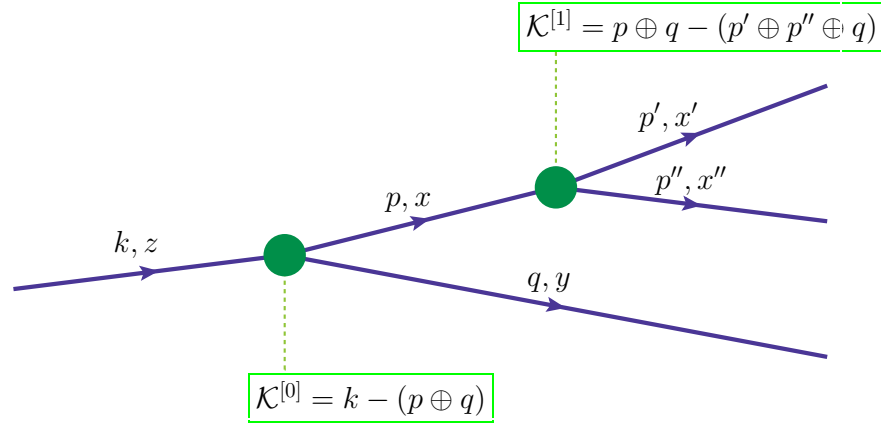


Figure 5. The case of causally-connected interactions analyzed in this subsection.

Of course, there is no difficulty generalizing to this case the bulk part of the action:

$$\begin{aligned} S_{bulk}^{\kappa(2)} &= \int_{-\infty}^{s_0} ds \left(z^\mu \dot{k}_\mu - \ell z^1 k_1 \dot{k}_0 + \mathcal{N}_k C_\kappa[k] \right) + \int_{s_0}^{s_1} ds \left(x^\mu \dot{p}_\mu - \ell x^1 p_1 \dot{p}_0 + \mathcal{N}_p C_\kappa[p] \right) \\ &+ \int_{s_1}^{+\infty} ds \left(x'^\mu \dot{p}'_\mu - \ell x'^1 p'_1 \dot{p}'_0 + \mathcal{N}_{p'} C_\kappa[p'] \right) + \int_{s_1}^{+\infty} ds \left(x''^\mu \dot{p}''_\mu - \ell x''^1 p''_1 \dot{p}''_0 + \mathcal{N}_{p''} C_\kappa[p''] \right) \\ &+ \int_{s_0}^{+\infty} ds \left(y^\mu \dot{q}_\mu - \ell y^1 q_1 \dot{q}_0 + \mathcal{N}_q C_\kappa[q] \right) . \end{aligned} \quad (86)$$

For the description of the interactions we take a case characterized by the following conservation laws:

$$k \oplus (\ominus q) \oplus (\ominus p) = 0 ,$$

$$p \oplus (\ominus p'') \oplus (\ominus p') = 0 .$$

And we propose an implementation of these conservation laws that is compatible with a relativistic description of distant observers, based on adding to the action constraints with

$$\mathcal{K}^{[0]} = k - (p \oplus q) ,$$

$$\mathcal{K}^{[1]} = (p \oplus q) - (p' \oplus p'' \oplus q) ,$$

i.e.

$$\begin{aligned} \mathcal{K}_\mu^{[0]} &= k_\mu - (p \oplus q)_\mu = k_\mu - p_\mu - q_\mu - \ell \delta_\mu^1 p_0 q_1 , \\ \mathcal{K}_\mu^{[1]} &= (p \oplus q)_\mu - (p' \oplus p'' \oplus q)_\mu = p_\mu - p'_\mu - p''_\mu - \ell \delta_\mu^1 (-p_0 q_1 + p'_0 p''_1 + p'_0 q_1 + p''_0 q_1) . \end{aligned} \quad (87)$$

For the constraints we are again implementing our prescription of writing them in terms of differences between the total momentum before the interaction and after the interaction. It may appear that in doing so we included in the constraints some irrelevant pieces (it is easy to verify that the conservation law $\mathcal{K}^{[1]} = 0$ is actually independent of q_μ , which is the momentum of the particle that is only a spectator of the interaction occurring at $s = s_1$). However, as we shall see, those extra pieces, while irrelevant for the physical content of the conservation laws, do play a role in the description of translation transformations and ensure the availability of a relativistic description of distant observers.

To show this let us then start by writing the full action for the two-interaction case on which we are presently focusing:

$$\begin{aligned} \mathcal{S}^{\kappa(2)} &= \mathcal{S}_{bulk}^{\kappa(2)} + \mathcal{S}_{int}^{\kappa(2)} = \int_{-\infty}^{s_0} ds (z^\mu \dot{k}_\mu - \ell z^1 k_1 \dot{k}_0 + \mathcal{N}_k C_\kappa[k]) + \int_{s_0}^{s_1} ds (x^\mu \dot{p}_\mu - \ell x^1 p_1 \dot{p}_0 + \mathcal{N}_p C_\kappa[p]) \\ &\quad + \int_{s_1}^{+\infty} ds (x'^\mu \dot{p}'_\mu - \ell x'^1 p'_1 \dot{p}'_0 + \mathcal{N}_{p'} C_\kappa[p']) + \int_{s_1}^{+\infty} ds (x''^\mu \dot{p}''_\mu - \ell x''^1 p''_1 \dot{p}''_0 + \mathcal{N}_{p''} C_\kappa[p'']) \\ &\quad + \int_{s_0}^{+\infty} ds (y^\mu \dot{q}_\mu - \ell y^1 q_1 \dot{q}_0 + \mathcal{N}_q C_\kappa[q]) \\ &\quad - \xi_{[0]}^\mu \mathcal{K}_\mu^{[0]}(s_0) - \xi_{[1]}^\mu \mathcal{K}_\mu^{[1]}(s_1) , \end{aligned} \quad (88)$$

where indeed with $\mathcal{K}^{[0]}$ and $\mathcal{K}^{[1]}$ we take respectively $k - (p \oplus q)$ and $(p \oplus q) - (p' \oplus p'' \oplus q)$.

It is again straightforward to derive the equations of motion (and constraints) that follow from our action $\mathcal{S}^{\kappa(2)}$:

$$\begin{aligned} \dot{p}_\mu &= 0 , \quad \dot{q}_\mu = 0 , \quad \dot{k}_\mu = 0 , \quad \dot{p}'_\mu = 0 , \quad \dot{p}''_\mu = 0 , \\ C_\kappa[p] &= 0 , \quad C_\kappa[q] = 0 , \quad C_\kappa[k] = 0 , \quad C_\kappa[p'] = 0 , \quad C_\kappa[p''] = 0 , \\ \mathcal{K}_\mu^{[0]}(s_0) &= 0 , \quad \mathcal{K}_\mu^{[1]}(s_1) = 0 , \end{aligned} \quad (89)$$

$$\begin{aligned} \dot{x}^\mu &= \mathcal{N}_p \left(\frac{\delta C_\kappa[p]}{\delta p_\mu} + \ell \delta_0^\mu \frac{\delta C_\kappa[p]}{\delta p_1} p_1 \right) = \delta_0^\mu \mathcal{N}_p (2p_0 - \ell p_1^2) - 2\delta_1^\mu \mathcal{N}_p (p_1 - \ell p_0 p_1) , \\ \dot{y}^\mu &= \mathcal{N}_q \left(\frac{\delta C_\kappa[q]}{\delta q_\mu} + \ell \delta_0^\mu \frac{\delta C_\kappa[q]}{\delta q_1} q_1 \right) = \delta_0^\mu \mathcal{N}_q (2q_0 - \ell q_1^2) - 2\delta_1^\mu \mathcal{N}_q (q_1 - \ell q_0 q_1) , \\ \dot{z}^\mu &= \mathcal{N}_k \left(\frac{\delta C_\kappa[k]}{\delta k_\mu} + \ell \delta_0^\mu \frac{\delta C_\kappa[k]}{\delta k_1} k_1 \right) = \delta_0^\mu \mathcal{N}_k (2k_0 - \ell k_1^2) - 2\delta_1^\mu \mathcal{N}_k (k_1 - \ell k_0 k_1) , \\ \dot{x}'^\mu &= \mathcal{N}_{p'} \left(\frac{\delta C_\kappa[p']}{\delta p'_\mu} + \ell \delta_0^\mu \frac{\delta C_\kappa[p']}{\delta p'_1} p'_1 \right) = \delta_0^\mu \mathcal{N}_{p'} (2p'_0 - \ell p'^2_1) - 2\delta_1^\mu \mathcal{N}_{p'} (p'_1 - \ell p'_0 p'_1) , \\ \dot{x}''^\mu &= \mathcal{N}_{p''} \left(\frac{\delta C_\kappa[p'']}{\delta p''_\mu} + \ell \delta_0^\mu \frac{\delta C_\kappa[p'']}{\delta p''_1} p''_1 \right) = \delta_0^\mu \mathcal{N}_{p''} (2p''_0 - \ell p''^2_1) - 2\delta_1^\mu \mathcal{N}_{p''} (p''_1 - \ell p''_0 p''_1) . \end{aligned} \quad (90)$$

And also the conditions at the $s = s_0$ and $s = s_1$ boundaries produced by the interaction terms are of rather standard relative-locality type:

$$\begin{aligned}
z^\mu(s_0) &= \xi_{[0]}^\nu \left(\frac{\delta \mathcal{K}_\nu^{[0]}}{\delta k_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_\nu^{[0]}}{\delta k_1} k_1 \right) = \xi_{[0]}^\mu + \ell \delta_0^\mu \xi_{[0]}^1 k_1, \\
x^\mu(s_0) &= -\xi_{[0]}^\nu \left(\frac{\delta \mathcal{K}_\nu^{[0]}}{\delta p_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_\nu^{[0]}}{\delta p_1} p_1 \right) = \xi_{[0]}^\mu + \ell \delta_0^\mu \xi_{[0]}^1 (p_1 + q_1), \quad x^\mu(s_1) = \xi_{[1]}^\nu \left(\frac{\delta \mathcal{K}_\nu^{[1]}}{\delta p_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_\nu^{[1]}}{\delta p_1} p_1 \right) = \xi_{[1]}^\mu + \ell \delta_0^\mu \xi_{[1]}^1 (p_1 + q_1), \\
y^\mu(s_0) &= -\xi_{[0]}^\nu \left(\frac{\delta \mathcal{K}_\nu^{[0]}}{\delta q_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_\nu^{[0]}}{\delta q_1} q_1 \right) = \xi_{[0]}^\mu + \ell \delta_0^\mu \xi_{[0]}^1 q_1 + \ell \delta_1^\mu \xi_{[0]}^1 p_0, \\
x^\mu(s_1) &= -\xi_{[1]}^\nu \left(\frac{\delta \mathcal{K}_\nu^{[1]}}{\delta p'_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_\nu^{[1]}}{\delta p'_1} p'_1 \right) = \xi_{[1]}^\mu + \ell \delta_0^\mu \xi_{[1]}^1 (p'_1 + p''_1 + q_1), \\
x''^\mu(s_1) &= -\xi_{[1]}^\nu \left(\frac{\delta \mathcal{K}_\nu^{[1]}}{\delta p''_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_\nu^{[1]}}{\delta p''_1} p''_1 \right) = \xi_{[1]}^\mu + \ell \delta_0^\mu \xi_{[1]}^1 (p''_1 + q_1) + \ell \delta_1^\mu \xi_{[1]}^1 p'_0.
\end{aligned} \tag{91}$$

However, thanks to our tailored choice of momentum-conservation constraints the boundary conditions at the two endpoints of the finite worldline exchanged by the two interactions (the finite worldline of particle coordinates $x^\mu(s)$ and momentum p_μ) match just in the right way to allow implementing as a relativistic symmetry the following translation transformations, generated by the total momentum

$$\begin{aligned}
z_B^0(s) &= z_A^0(s) + b^\mu \{k_\mu, z^0\} = z_A^0(s) - b^0 - \ell b^1 k_1, \\
z_B^1(s) &= z_A^1(s) + b^\mu \{k_\mu, z^1\} = z_A^1(s) - b^1, \\
x_B^0(s) &= x_A^0(s) + b^\mu \{(p \oplus q)_\mu, x^0\} = x_A^0(s) - b^0 - \ell b^1 (p_1 + q_1), \\
x_B^1(s) &= x_A^1(s) + b^\mu \{(p \oplus q)_\mu, x^0\} = x_A^1(s) - b^1, \\
y_B^0(s) &= y_A^0(s) + b^\mu \{(p \oplus q)_\mu, y^0\} = y_A^0(s) - b^0 - \ell b^1 q_1, \\
y_B^1(s) &= y_A^1(s) + b^\mu \{(p \oplus q)_\mu, y^1\} = y_A^1(s) - b^1 - \ell b^1 p_0, \\
x_B'^0(s) &= x_A'^0(s) + b^\mu \{(p' \oplus p'' \oplus q)_\mu, x'^0\} = x_A'^0(s) - b^0 - \ell b^1 (p'_1 + p''_1 + q_1), \\
x_B'^1(s) &= x_A'^1(s) + b^\mu \{(p' \oplus p'' \oplus q)_\mu, x'^1\} = x_A'^1(s) - b^1, \\
x_B''^0(s) &= x_A''^0(s) + b^\mu \{(p' \oplus p'' \oplus q)_\mu, x''^0\} = x_A''^0(s) - b^0 - \ell b^1 (p''_1 + q_1), \\
x_B''^1(s) &= x_A''^1(s) + b^\mu \{(p' \oplus p'' \oplus q)_\mu, x''^1\} = x_A''^1(s) - b^1 - \ell b^1 p'_0.
\end{aligned} \tag{92}$$

It is again straightforward to see that these transformations leave the equations of motion (90) unchanged by noticing, as done in the previous subsection, that the only non trivial terms in the deformed translations (92) depend on momenta and the momenta are conserved along the worldlines.

And it is also easy to verify that our translation transformations leave the boundary conditions unchanged. In order to give an explicit example let us check the case of x''^μ : substituting the translation calculated in Eq. (92)

$$\begin{aligned}
x_B''^0(s) &= x_A''^0 - b^0 - \ell b^1 (p'_1 + q_1), \\
x_B''^1(s) &= x_A''^1 - b^1 - \ell b^1 p'_0, \\
\xi_B^\mu &= \xi_A^\mu - b^\mu,
\end{aligned}$$

in the boundary conditions (91)

$$\begin{aligned}
x_B''^0(s_1) &= -\xi_{[1]B}^\nu \left(\frac{\delta \mathcal{K}_\nu^{[1]}}{\delta p''_0} + \ell \frac{\delta \mathcal{K}_\nu^{[1]}}{\delta p''_1} p''_1 \right) = \xi_{[1]B}^0 + \ell \xi_{[1]B}^1 (q_1 + p''_1), \\
x_B''^1(s_1) &= -\xi_{[1]B}^\nu \left(\frac{\delta \mathcal{K}_\nu^{[1]}}{\delta p''_1} \right) = \xi_{[1]B}^1 (1 + \ell p'_0),
\end{aligned}$$

we find

$$\begin{aligned}
x_B''^0(s_1) - \xi_{[1]B}^0 - \ell \xi_{[1]B}^1 (q_1 + p''_1) &= x_A''^0(s_1) - \xi_{[1]A}^0 - \ell \xi_{[1]A}^1 (q_1 + p''_1), \\
x_B''^1(s_1) - \xi_{[1]B}^1 (1 + \ell p'_0) &= x_A''^1(s_1) - \xi_{[1]A}^1 (1 + \ell p'_0).
\end{aligned}$$

which is evidently consistent with our boundary conditions.

Thus we did succeed: even in the case of finite worldlines, causally connecting pairs of interactions, our prescription for boundary terms does ensure translational invariance, addressing the challenge highlighted here in Section V D. And our relative-locality distant observers are connected by relativistic transformations generated by the (\oplus -deformed) total momentum.

The invariance of the equations of motion and boundary conditions under our translations generated by the total momentum is also manifest in the properties of the action under these translation transformations. In fact, it turns out that these translation transformations do change the action $S^{\kappa(2)}$, but only by terms that do not contribute to the equations of motion (once the constraints are taken into account). In order to see this explicitly let us start by noticing that we can split the integral for the worldline p, x in (88) in the following way:

$$\int_{s_0}^{s_1} ds (x^\mu \dot{p}_\mu - \ell x^1 \dot{p}_0 p_1 + \mathcal{N}_p C_\kappa[p]) = \int_{s_0}^\infty ds (x^\mu \dot{p}_\mu - \ell x^1 \dot{p}_0 p_1 + \mathcal{N}_p C_\kappa[p]) - \int_{s_1}^\infty ds (x^\mu \dot{p}_\mu - \ell x^1 \dot{p}_0 p_1 + \mathcal{N}_p C_\kappa[p]) .$$

So we can separate in the action (88) the contributions relative to the interactions at s_0 and s_1 (contributions with boundary at s_0 and contributions with boundary at s_1). The part relative to the vertex s_0 is the same as the action (73) analyzed in the previous section. We consider then only the contributions with boundary at s_1 :

$$\begin{aligned} \Delta S_{s_1}^{\kappa(2)} = & - \int_{s_1}^\infty ds (-b^\mu \dot{p}_\mu - \ell b^1 q_1 \dot{p}_0) + \int_{s_1}^\infty ds (-b^\mu \dot{p}'_\mu - \ell b^1 p_1'' \dot{p}'_0 - \ell b^1 q_1 \dot{p}'_0) \\ & + \int_{s_1}^\infty ds (-b^\mu \dot{p}''_\mu - \ell b^1 q_1 \dot{p}''_0 - \ell b^1 p_0' \dot{p}''_1) + \Delta \xi_{[1]}^\mu \mathcal{K}_\mu^{[1]}(s_1) . \end{aligned}$$

This evidently can be rewritten as

$$\Delta S_{s_1}^{\kappa(2)} = \int_{s_1}^\infty ds (b^\mu (\dot{p}_\mu - \dot{p}'_\mu - \dot{p}''_\mu) - \ell b^1 (-q_1 \dot{p}_0 + p_1'' \dot{p}'_0 + q_1 \dot{p}'_0 + q_1 \dot{p}''_0 + p_0' \dot{p}''_1)) + \Delta \xi_{[1]}^\mu \mathcal{K}_\mu^{[1]}(s_1) ,$$

which, taking into account Eq. (87), gives

$$\Delta S_{s_1}^{\kappa(2)} = \int_{s_1}^\infty ds \left(\frac{d}{ds} [b^\mu \mathcal{K}_\mu^{[1]}] - \ell b^1 \mathcal{K}_0^{[1]} \dot{q}_1 \right) + \Delta \xi_{[1]}^\mu \mathcal{K}_\mu^{[1]}(s_1) .$$

The total derivative contributes as before to the translation of $\xi_{[1]}^\mu$: $\xi_{[1]B}^\mu = \xi_{[1]A}^\mu - b^\mu$. In addition there is a left over bulk term,

$$\int_{s_1}^\infty ds \ell b^1 \mathcal{K}_0^{[1]} \dot{q}_1 ,$$

but it is evidently inconsequential for what concerns the equations of motion. In fact, varying this left-over term one finds

$$\int_{s_1}^\infty ds \ell b^1 (\delta \mathcal{K}_0^{[1]} \dot{q}_1 + \mathcal{K}_0^{[1]} \delta \dot{q}_1) ,$$

i.e.

$$\int_{s_1}^\infty ds \ell b^1 \left(\delta \mathcal{K}_0^{[1]} \dot{q}_1 - \dot{\mathcal{K}}_0^{[1]} \delta q_1 + \frac{d}{ds} \mathcal{K}_0^{[1]} \delta q_1 \right) .$$

And the 3 terms in this expression contribute to equations of motion and boundary conditions only terms which are already fixed to vanish because of constraints derived from other parts of the action (specifically $\dot{p}_\mu = 0$, $\dot{q}_\mu = 0$, $\dot{p}'_\mu = 0$, $\dot{p}''_\mu = 0$ and $\mathcal{K}_0^{[1]} = 0$).

C. Aside on an alternative choice of action

In the previous subsection we showed that causally-connected interactions, with relative locality, can be formulated consistently with translational invariance, and therefore admit a relativistic description of distant observers. Crucial for our result was noticing that the equations of motion and the boundary conditions are invariant under our proposed translation transformations, generated by the total momentum, even though those translation transformations did not leave the action unchanged in the bulk. For completeness in this subsection we want to show that exactly the same physical proposal of the previous subsection can be given in terms of a different action, with slightly different boundary terms (at endpoints of worldlines).

Ultimately the difference between the two alternatives we shall then have amounts to the properties of the two actions under the same laws of translation transformation: the case in the previous subsection was such that translation transformations changed the action in the bulk (but without changing the equations of motion), while the case we discuss in this subsection will turn out to be such that translation transformations change the action on the boundary, but without affecting the boundary conditions. The two actions give exactly the same physical picture.

We consider exactly the same configuration already analyzed in the previous subsection, but (as hinted at in Fig. 6) in addition to the constraints given in terms of

$$\mathcal{K}^{[0]}(s_0) = k - (p \oplus q) ,$$

$$\mathcal{K}^{[1]}(s_1) = (p \oplus q) - (p' \oplus p'' \oplus q) ,$$

we add, as a technical expedient, another interaction also at $s = 1$, a bivalent interaction (a non-interaction) characterized by the a constraint given in terms of

$$\mathcal{K}^{[1']}(s_1) = (p \oplus q) - (p \oplus q^*) .$$

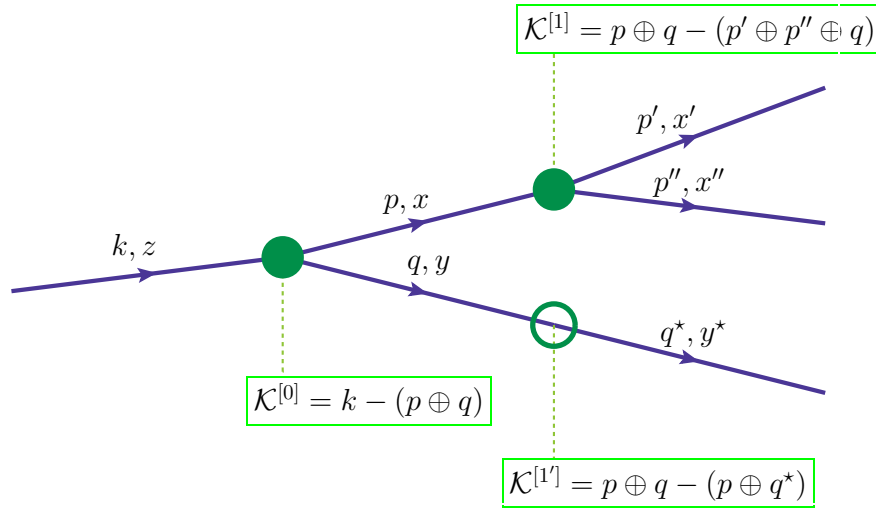


Figure 6. Our choices of boundary terms for a pair of causally-connected interactions, when using the expedient of a bivalent interaction in combination with the second trivalent interaction.

The added (fictitious) bivalent interaction leads to replacing the action $S^{\kappa(2)}$ with

$$\begin{aligned} S^{\kappa(2')} = & \int_{-\infty}^{s_0} ds (z^\mu \dot{k}_\mu - \ell z^1 k_1 \dot{k}_0 + \mathcal{N}_k C_\kappa[k]) + \int_{s_0}^{s_1} ds (x^\mu \dot{p}_\mu - \ell x^1 p_1 \dot{p}_0 + \mathcal{N}_p C_\kappa[p]) \\ & + \int_{s_0}^{s_1} ds (y^\mu \dot{q}_\mu - \ell y^1 q_1 \dot{q}_0 + \mathcal{N}_q C_\kappa[q]) + \int_{s_1}^{\infty} ds (y_\star^\mu \dot{q}_\mu^\star - \ell y_\star^1 q_1^\star \dot{q}_0^\star + \mathcal{N}_{q^\star} C_\kappa[q^\star]) \\ & + \int_{s_1}^{\infty} ds (x'^\mu \dot{p}'_\mu - \ell x'^1 p'_1 \dot{p}'_0 + \mathcal{N}_{p'} C_\kappa[p']) + \int_{s_1}^{\infty} ds (x''^\mu \dot{p}''_\mu - \ell x''^1 p''_1 \dot{p}''_0 + \mathcal{N}_{p''} C_\kappa[p'']) \\ & - \xi_{[0]}^\mu \mathcal{K}_\mu^{[0]}(s_0) - \xi_{[1]}^\mu \mathcal{K}_\mu^{[1]}(s_1) - \xi_{[1']}^\mu \mathcal{K}_\mu^{[1']}(s_1) . \end{aligned} \quad (93)$$

In Appendix A we show that this action produces exactly the same equations of motion and boundary conditions as the action considered in the previous subsection, with only peculiarity that (as suggested by the drawing in Fig. 6) the worldline y^μ , q_μ of the previous subsection gets here fictitiously split into two perfectly-matching pieces of worldline, a piece labeled again y^μ , q_μ and a piece labeled y_\star^μ , q_μ^\star .

The same applies for our description of translation transformations generated by the total momentum, which for the action

$S^{\kappa(2')}$ takes the form

$$\begin{aligned}
z_B^0(s) &= z_A^0(s) + b^\mu \{k_\mu, z^0\} = z_A^0(s) - b^0 - \ell b^1 k_1, \\
z_B^1(s) &= z_A^1(s) + b^\mu \{k_\mu, z^1\} = z_A^1(s) - b^1, \\
x_B^0(s) &= x_A^0(s) + b^\mu \{(p \oplus q)_\mu, x^0\} = x_A^0(s) - b^0 - \ell b^1 (p_1 + q_1), \\
x_B^1(s) &= x_A^1(s) + b^\mu \{(p \oplus q)_\mu, x^1\} = x_A^1(s) - b^1, \\
y_B^0(s) &= y_A^0(s) + b^\mu \{(p \oplus q)_\mu, y^0\} = y_A^0(s) - b^0 - \ell b^1 q_1, \\
y_B^1(s) &= y_A^1(s) + b^\mu \{(p \oplus q)_\mu, y^1\} = y_A^1(s) - b^1 - \ell b^1 p_0, \\
y_B^{\star 0}(s) &= y_A^{\star 0}(s) + b^\mu \{(p' \oplus p'' \oplus q^\star)_\mu, y^{\star 0}\} = y_A^{\star 0}(s) - b^0 - \ell b^1 q_1^\star, \\
y_B^{\star 1}(s) &= y_A^{\star 1}(s) + b^\mu \{(p' \oplus p'' \oplus q^\star)_\mu, y^{\star 1}\} = y_A^{\star 1}(s) - b^1 - \ell b^1 (p'_0 + p''_0), \\
x_B^{\prime 0}(s) &= x_A^{\prime 0}(s) + b^\mu \{(p' \oplus p'' \oplus q^\star)_\mu, x^{\prime 0}\} = x_A^{\prime 0}(s) - b^0 - \ell b^1 (p'_1 + p''_1 + q_1^\star), \\
x_B^{\prime 1}(s) &= x_A^{\prime 1}(s) + b^\mu \{(p' \oplus p'' \oplus q^\star)_\mu, x^{\prime 1}\} = x_A^{\prime 1}(s) - b^1, \\
x_B^{\prime\prime 0}(s) &= x_A^{\prime\prime 0}(s) + b^\mu \{(p' \oplus p'' \oplus q^\star)_\mu, x^{\prime\prime 0}\} = x_A^{\prime\prime 0}(s) - b^0 - \ell b^1 (p''_1 + q_1^\star), \\
x_B^{\prime\prime 1}(s) &= x_A^{\prime\prime 1}(s) + b^\mu \{(p' \oplus p'' \oplus q^\star)_\mu, x^{\prime\prime 1}\} = x_A^{\prime\prime 1}(s) - b^1 - \ell b^1 p'_0.
\end{aligned} \tag{94}$$

These translation transformations are symmetries of the equations of motion and boundary conditions (given in Appendix A), and they essentially are the same translation transformations we discussed in the previous subsection, up to splitting again fictitiously the worldline y^μ, q_μ into pieces y^μ, q_μ and y_\star^μ, q_μ^\star .

It is also easy to check that the action $S^{\kappa(2')}$ does change under this translation transformations, but only by an amount that can be expressed in terms of other boundary terms. Indeed repeating the same steps as before, i.e. substituting the relations (94) in the action (93), and repeating the algebraic manipulations showed previously, we find

$$S_B^{\kappa(2')} = S_A^{\kappa(2')} + \ell b^1 \mathcal{K}_0^{[1]} \mathcal{K}_1^{[1']}. \tag{95}$$

And it is particularly clear that this additional boundary term for the “translated action” is irrelevant: it has no implication on the boundary conditions (it would produce additional boundary conditions which however are automatically satisfied once the other boundary conditions are enforced).

D. The case of 3 connected finite worldlines

At this point we have established that at least in the simplest applications our prescriptions do provide the desired relativistic picture. In order to motivate our next consistency check it is useful to look at available results on relative locality from the following perspective:

- ★ with the Hamiltonian description of relative locality for free particles given in Refs. [15–18] one essentially obtains a characterization of relative locality limited to infinite worldlines;
- ★ with the Lagrangian description of relative locality for interacting particles proposed in Ref. [1] the availability of a relativistic description of distant observers had been checked explicitly only for semi-infinite worldlines (a single interaction);
- ★ the results reported so far in this section generalize the results for distant observers of Ref. [1] to the case where one of the worldlines is finite (a worldline exchanged between two interactions, establishing the causal relation between the two interactions).

In this subsection we provide evidence of the fact that our prescription is robust also for cases with several finite worldlines. We actually consider here a case which is very meaningful from this perspective: the case shown in Figure 7, which includes a vertex where 3 finite worldlines meet.

Following the prescription we are advocating the situation in Figure 7 requires handling boundary terms with

$$\begin{aligned}
\mathcal{K}^{[0]} &= k - k' \oplus k'', \\
\mathcal{K}^{[1]} &= k' \oplus k'' - p \oplus q \oplus k'', \\
\mathcal{K}^{[2]} &= p \oplus q \oplus k'' - p' \oplus p'' \oplus q \oplus k'', \\
\mathcal{K}^{[3]} &= p' \oplus p'' \oplus q \oplus k'' - p' \oplus p'' \oplus q' \oplus q'' \oplus k''.
\end{aligned} \tag{96}$$

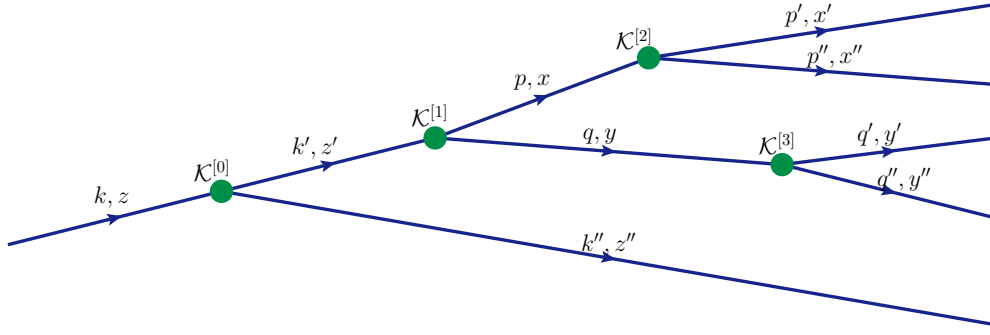


Figure 7. The case with 3 connected finite worldlines which we consider in this subsection.

We therefore describe the chain of interactions in Figure 7 through the following action:

$$\begin{aligned}
\mathcal{S}^{(3conn)} = & \int_{-\infty}^{s_0} ds (z^\mu \dot{k}_\mu - \ell z^1 k_1 \dot{k}_0 + \mathcal{N}_k C[k]) + \int_{s_0}^{+\infty} ds (z''^\mu \dot{k}_\mu'' - \ell z''^1 k_1'' \dot{k}_0'' + \mathcal{N}_{k''} C[k]) \\
& + \int_{s_0}^{s_1} ds (z'^\mu \dot{k}_\mu' - \ell z'^1 k_1' \dot{k}_0' + \mathcal{N}_{k'} C[k']) + \int_{s_1}^{s_2} ds (x^\mu \dot{p}_\mu - \ell x^1 p_1 \dot{p}_0 + \mathcal{N}_p C[p]) \\
& + \int_{s_2}^{+\infty} ds (x''^\mu \dot{p}_\mu'' - \ell x''^1 p_1'' \dot{p}_0'' + \mathcal{N}_{p''} C[p'']) + \int_{s_2}^{+\infty} ds (x'^\mu \dot{p}_\mu' - \ell x'^1 p_1' \dot{p}_0' + \mathcal{N}_{p'} C[p']) \\
& + \int_{s_1}^{s_3} ds (y^\mu \dot{q}_\mu - \ell y^1 q_1 \dot{q}_0 + \mathcal{N}_q C[q]) + \int_{s_3}^{+\infty} ds (y''^\mu \dot{q}_\mu'' - \ell y''^1 q_1'' \dot{q}_0'' + \mathcal{N}_{q''} C[q'']) \\
& + \int_{s_3}^{+\infty} ds (y'^\mu \dot{q}_\mu' - \ell y'^1 q_1' \dot{q}_0' + \mathcal{N}_{q'} C[q']) \\
& - \xi_{[0]}^\mu \mathcal{X}_\mu^{[0]}(s_0) - \xi_{[1]}^\mu \mathcal{X}_\mu^{[1]}(s_1) - \xi_{[2]}^\mu \mathcal{X}_\mu^{[2]}(s_2) - \xi_{[3]}^\mu \mathcal{X}_\mu^{[3]}(s_3) .
\end{aligned} \tag{97}$$

For what concerns equations of motion and boundary conditions we have

$$\begin{aligned}
\dot{p}_\mu = 0, \quad \dot{p}'_\mu = 0, \quad \dot{p}''_\mu = 0, \quad \dot{q}_\mu = 0, \quad \dot{q}'_\mu = 0, \quad \dot{q}''_\mu = 0, \quad \dot{k}_\mu = 0, \quad \dot{k}'_\mu = 0, \quad \dot{k}''_\mu = 0, \\
C_\kappa[p] = 0, \quad C_\kappa[p'] = 0, \quad C_\kappa[p''] = 0, \quad C_\kappa[q] = 0, \quad C_\kappa[q'] = 0, \quad C_\kappa[q''] = 0, \quad C_\kappa[k] = 0, \quad C_\kappa[k'] = 0, \quad C_\kappa[k''] = 0,
\end{aligned}$$

$$\begin{aligned}
\dot{x}^\mu - \mathcal{N}_p \left(\frac{\delta C_\kappa[p]}{\delta p_\mu} + \ell \delta_0^\mu \frac{\delta C_\kappa[p]}{\delta p_1} p_1 \right) = 0, \quad \dot{x}'^\mu - \mathcal{N}_{p'} \left(\frac{\delta C_\kappa[p']}{\delta p'_\mu} + \ell \delta_0^\mu \frac{\delta C_\kappa[p']}{\delta p'_1} p'_1 \right) = 0, \quad \dot{x}''^\mu - \mathcal{N}_{p''} \left(\frac{\delta C_\kappa[p'']}{\delta p''_\mu} + \ell \delta_0^\mu \frac{\delta C_\kappa[p'']}{\delta p''_1} p''_1 \right) = 0, \\
\dot{y}^\mu - \mathcal{N}_q \left(\frac{\delta C_\kappa[q]}{\delta q_\mu} + \ell \delta_0^\mu \frac{\delta C_\kappa[q]}{\delta q_1} q_1 \right) = 0, \quad \dot{y}'^\mu - \mathcal{N}_{q'} \left(\frac{\delta C_\kappa[q']}{\delta q'_\mu} + \ell \delta_0^\mu \frac{\delta C_\kappa[q']}{\delta q'_1} q'_1 \right) = 0, \quad \dot{y}''^\mu - \mathcal{N}_{q''} \left(\frac{\delta C_\kappa[q'']}{\delta q''_\mu} + \ell \delta_0^\mu \frac{\delta C_\kappa[q'']}{\delta q''_1} q''_1 \right) = 0, \\
\dot{z}^\mu - \mathcal{N}_k \left(\frac{\delta C_\kappa[k]}{\delta k_\mu} + \ell \delta_0^\mu \frac{\delta C_\kappa[k]}{\delta k_1} k_1 \right) = 0, \quad \dot{z}'^\mu - \mathcal{N}_{k'} \left(\frac{\delta C_\kappa[k']}{\delta k'_\mu} + \ell \delta_0^\mu \frac{\delta C_\kappa[k']}{\delta k'_1} k'_1 \right) = 0, \quad \dot{z}''^\mu - \mathcal{N}_{k''} \left(\frac{\delta C_\kappa[k'']}{\delta k''_\mu} + \ell \delta_0^\mu \frac{\delta C_\kappa[k'']}{\delta k''_1} k''_1 \right) = 0,
\end{aligned}$$

$$\begin{aligned}
z^\mu(s_0) &= \xi_{[0]}^v \left(\frac{\delta \mathcal{K}_v^{[0]}}{\delta k_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_v^{[0]}}{\delta k_1} k_1 \right), \\
z'^\mu(s_0) &= -\xi_{[0]}^v \left(\frac{\delta \mathcal{K}_v^{[0]}}{\delta k'_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_v^{[0]}}{\delta k'_1} k'_1 \right), \quad z'^\mu(s_1) = \xi_{[1]}^v \left(\frac{\delta \mathcal{K}_v^{[1]}}{\delta k'_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_v^{[1]}}{\delta k'_1} k'_1 \right), \\
z''^\mu(s_0) &= -\xi_{[0]}^v \left(\frac{\delta \mathcal{K}_v^{[0]}}{\delta k''_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_v^{[0]}}{\delta k''_1} k''_1 \right), \\
x^\mu(s_1) &= -\xi_{[1]}^v \left(\frac{\delta \mathcal{K}_v^{[1]}}{\delta p_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_v^{[1]}}{\delta p_1} p_1 \right), \quad x^\mu(s_2) = \xi_{[2]}^v \left(\frac{\delta \mathcal{K}_v^{[2]}}{\delta p_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_v^{[2]}}{\delta p_1} p_1 \right), \\
y^\mu(s_1) &= -\xi_{[1]}^v \left(\frac{\delta \mathcal{K}_v^{[1]}}{\delta q_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_v^{[1]}}{\delta q_1} q_1 \right), \quad y^\mu(s_3) = \xi_{[3]}^v \left(\frac{\delta \mathcal{K}_v^{[3]}}{\delta q_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_v^{[3]}}{\delta q_1} q_1 \right), \\
x'^\mu(s_2) &= -\xi_{[2]}^v \left(\frac{\delta \mathcal{K}_v^{[2]}}{\delta p'_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_v^{[2]}}{\delta p'_1} p'_1 \right), \\
x''^\mu(s_2) &= -\xi_{[2]}^v \left(\frac{\delta \mathcal{K}_v^{[2]}}{\delta p''_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_v^{[2]}}{\delta p''_1} p''_1 \right), \\
y'^\mu(s_3) &= -\xi_{[3]}^v \left(\frac{\delta \mathcal{K}_v^{[3]}}{\delta q'_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_v^{[3]}}{\delta q'_1} q'_1 \right), \\
y''^\mu(s_3) &= -\xi_{[3]}^v \left(\frac{\delta \mathcal{K}_v^{[3]}}{\delta q''_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_v^{[3]}}{\delta q''_1} q''_1 \right).
\end{aligned}$$

And our notion of translation transformation to a distant observer is such that

$$\begin{aligned}
x_B^0 &= x_A^0 + b^\mu \{ (p \oplus q \oplus k'')_\mu, x^0 \} = x_A^0(s) - b^0 - \ell b^1 (p_1 + q_1 + k'_1), \\
x_B^1 &= x_A^1 + b^\mu \{ (p \oplus q \oplus k'')_\mu, x^1 \} = x_A^1(s) - b^1, \\
x_B'^0 &= x_A'^0 + b^\mu \{ (p' \oplus p'' \oplus q \oplus k'')_\mu, x'^0 \} = x_A'^0(s) - b^0 - \ell b^1 (p'_1 + p''_1 + q_1 + k'_1), \\
x_B'^1 &= x_A'^1 + b^\mu \{ (p' \oplus p'' \oplus q \oplus k'')_\mu, x'^1 \} = x_A'^1(s) - b^1, \\
x_B''^0 &= x_A''^0 + b^\mu \{ (p' \oplus p'' \oplus q \oplus k'')_\mu, x''^0 \} = x_A''^0(s) - b^0 - \ell b^1 (p''_1 + q_1 + k'_1), \\
x_B''^1 &= x_A''^1 + b^\mu \{ (p' \oplus p'' \oplus q \oplus k'')_\mu, x''^1 \} = x_A''^1(s) - b^1 - \ell b^1 p'_0, \\
y_B^0 &= y_A^0 + b^\mu \{ (p \oplus q \oplus k'')_\mu, y^0 \} = y_A^0(s) - b^0 - \ell b^1 (q_1 + k'_1), \\
y_B^1 &= y_A^1 + b^\mu \{ (p \oplus q \oplus k'')_\mu, y^1 \} = y_A^1(s) - b^1 - \ell b^1 p_0, \\
y_B'^0 &= y_A'^0 + b^\mu \{ (p' \oplus p'' \oplus q' \oplus q'' \oplus k'')_\mu, y'^0 \} = y_A'^0(s) - b^0 - \ell b^1 (q'_1 + q''_1 + k'_1), \\
y_B'^1 &= y_A'^1 + b^\mu \{ (p' \oplus p'' \oplus q' \oplus q'' \oplus k'')_\mu, y'^1 \} = y_A'^1(s) - b^1 - \ell b^1 (p'_0 + p''_0), \\
y_B''^0 &= y_A''^0 + b^\mu \{ (p' \oplus p'' \oplus q' \oplus q'' \oplus k'')_\mu, y''^0 \} = y_A''^0(s) - b^0 - \ell b^1 (q'_1 + k'_1), \\
y_B''^1 &= y_A''^1 + b^\mu \{ (p' \oplus p'' \oplus q' \oplus q'' \oplus k'')_\mu, y''^1 \} = y_A''^1(s) - b^1 - \ell b^1 (p'_0 + p''_0 + q'_0), \\
z_B^0 &= z_A^0 + b^\mu \{ k_\mu, z^0 \} = z_A^0(s) - b^0 - \ell b^1 k_1, \\
z_B^1 &= z_A^1 + b^\mu \{ k_\mu, z^1 \} = z_A^1(s) - b^1, \\
z_B'^0 &= z_A'^0 + b^\mu \{ (k' \oplus k'')_\mu, z'^0 \} = z_A'^0(s) - b^0 - \ell b^1 (k'_1 + k''_1), \\
z_B'^1 &= z_A'^1 + b^\mu \{ (k' \oplus k'')_\mu, z'^1 \} = z_A'^1(s) - b^1, \\
z_B''^0 &= z_A''^0 + b^\mu \{ (k' \oplus k'')_\mu, z''^0 \} = z_A''^0(s) - b^0 - \ell b^1 k''_1, \\
z_B''^1 &= z_A''^1 + b^\mu \{ (k' \oplus k'')_\mu, z''^1 \} = z_A''^1(s) - b^1 - \ell b^1 k'_0.
\end{aligned} \tag{98}$$

It is then easy to check that also in this case the equations of motion and boundary conditions are left unchanged by our notion of translation transformation to a distant observer. This is also verifiable by studying the implications of our translation transformations for the action $\mathcal{S}^{(3\text{conn})}$, to which we devote Appendix B.

VIII. IMPLICATIONS FOR THE TIMES OF ARRIVAL OF SIMULTANEOUSLY-EMITTED ULTRARELATIVISTIC PARTICLES

In the previous section we established the basic notions and key characterizing results of our proposal of a first example of prescriptions for boundary terms ensuring a relativistic description of distant observers within the relative-locality framework of Ref. [1], with a lagrangian formulation of interacting particles.

In this section we extend the scopes of our analysis slightly beyond basics, by focusing on a first point of phenomenological relevance, concerning observations of distant bursts of massless particles.

In the process we shall also show that there is an appropriate limit where our more powerful formalism reproduces the previous results (here reviewed in Section VI) of the Hamiltonian description of free κ -Minkowski particles.

A. Matching Lagrangian and Hamiltonian description of κ -Minkowski free particles

Let us indeed start this section by showing that our proposal for translation transformations, besides fulfilling the demands of relativistic consistency verified in the previous section, also has the welcome property of reproducing the previous results (here reviewed in Section VI) of the Hamiltonian description of free κ -Poincaré particles. Of course this occurs in an appropriate limit of our framework, since in general our framework describes interacting κ -Poincaré particles. A key observation from this perspective is that a particle is still “essentially free” when its interactions only involve exchanges of very small fractions of its momentum.

As an illustrative example of a situation where these concepts apply and the mentioned “free Hamiltonian limit” is matched, we consider the situation shown in Figure 8.

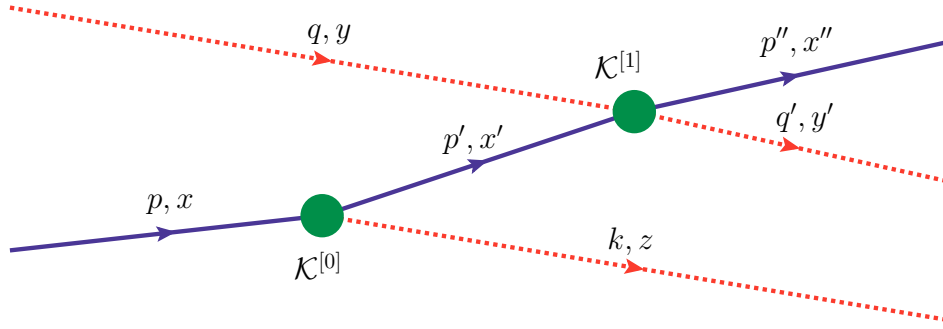


Figure 8. Schematics of a pion decaying into a soft and a hard photon, with the hard photon ultimately detected through an interaction in which it exchanges a small part of its momentum with a particle in a detector (hard worldlines in solid blue, soft worldlines in dotted red)

Notice that the situation in Figure 8 is also relevant for the description of observations of gamma-ray bursts: the incoming blue worldline p, x could be, *e.g.*, a highly boosted pion, which decays at the source, producing a gamma ray (p', x') and a very soft photon (k, z); then the gamma ray propagates freely until its first interaction at the detector, where it exchanges a small amount of momentum with a soft particle (q, y). So we can ask if and how the time of detection of the gamma ray depends on its momentum p' ; thereby obtaining a prediction for the large class of studies which is considering possible energy/time-of-arrival correlations for observations of gamma-ray bursts (see, *e.g.*, Refs. [33–36]).

An action which is suitable for the relative-locality description of the process shown in Figure 8 is

$$\begin{aligned}
 \mathcal{S}^{\kappa(2)} = & \int_{s_0}^{+\infty} ds (z^\mu \dot{k}_\mu - \ell z^1 k_1 \dot{k}_0 + \mathcal{N}_k C_\kappa[k]) + \int_{-\infty}^{s_0} ds (x^\mu \dot{p}_\mu - \ell x^1 p_1 \dot{p}_0 + \mathcal{N}_p C_\kappa[p]) \\
 & + \int_{s_0}^{s_1} ds (x'^\mu \dot{p}'_\mu - \ell x'^1 p'_1 \dot{p}'_0 + \mathcal{N}_{p'} C_\kappa[p']) + \int_{s_1}^{+\infty} ds (x''^\mu \dot{p}''_\mu - \ell x''^1 p''_1 \dot{p}''_0 + \mathcal{N}_{p''} C_\kappa[p'']) \\
 & + \int_{-\infty}^{s_1} ds (y^\mu \dot{q}_\mu - \ell y^1 q_1 \dot{q}_0 + \mathcal{N}_q C_\kappa[q]) + \int_{s_1}^{+\infty} ds (y'^\mu \dot{q}'_\mu - \ell y'^1 q'_1 \dot{q}'_0 + \mathcal{N}_{q'} C_\kappa[q']) \\
 & - \xi_{[0]}^\mu \mathcal{X}_{\mu}^{[0]}(s_0) - \xi_{[1]}^\mu \mathcal{X}_{\mu}^{[1]}(s_1) ,
 \end{aligned} \tag{99}$$

where

$$\begin{aligned}\mathcal{K}_\mu^{[0]}(s_0) &= (q \oplus p)_\mu - (q \oplus p' \oplus k)_\mu = p_\mu - p'_\mu - k_\mu - \ell \delta_\mu^1 (-q_0 p_1 + q_0 p'_1 + q_0 k'_1 + p'_0 k_1), \\ \mathcal{K}_\mu^{[1]}(s_1) &= (q \oplus p' \oplus k)_\mu - (p'' \oplus q' \oplus k)_\mu = q_\mu + p'_\mu - p''_\mu - q'_\mu - \ell \delta_\mu^1 (-q_0 p'_1 - q_0 k_1 - p'_0 k_1 + p''_0 q'_1 + p''_0 k_1 + q'_0 k_1).\end{aligned}\quad (100)$$

And, following the procedure we already used several times, from this action one obtains easily the equations of motion and the constraints,

$$\begin{aligned}\dot{p}_\mu &= 0, \quad \dot{q}_\mu = 0, \quad \dot{k}_\mu = 0, \quad \dot{p}'_\mu = 0, \quad \dot{p}''_\mu = 0, \\ C_\kappa[p] &= 0, \quad C_\kappa[q] = 0, \quad C_\kappa[k] = 0, \quad C_\kappa[p'] = 0, \quad C_\kappa[p''] = 0, \\ z^\mu - \mathcal{N}_k \left(\frac{\delta C_\kappa[k]}{\delta k_\mu} + \ell \delta_0^\mu \frac{\delta C_\kappa[k]}{\delta k_1} k_1 \right) &= 0, \quad y^\mu - \mathcal{N}_q \left(\frac{\delta C_\kappa[q]}{\delta q_\mu} + \ell \delta_0^\mu \frac{\delta C_\kappa[q]}{\delta q_1} q_1 \right) = 0, \\ y'^\mu - \mathcal{N}_{q'} \left(\frac{\delta C_\kappa[q']}{\delta q'_\mu} + \ell \delta_0^\mu \frac{\delta C_\kappa[q']}{\delta q'_1} q'_1 \right) &= 0, \quad x^\mu - \mathcal{N}_p \left(\frac{\delta C_\kappa[p]}{\delta p_\mu} + \ell \delta_0^\mu \frac{\delta C_\kappa[p]}{\delta p_1} p_1 \right) = 0, \\ x'^\mu - \mathcal{N}_{p'} \left(\frac{\delta C_\kappa[p']}{\delta p'_\mu} + \ell \delta_0^\mu \frac{\delta C_\kappa[p']}{\delta p'_1} p'_1 \right) &= 0, \quad x''^\mu - \mathcal{N}_{p''} \left(\frac{\delta C_\kappa[p'']}{\delta p''_\mu} + \ell \delta_0^\mu \frac{\delta C_\kappa[p'']}{\delta p''_1} p''_1 \right) = 0,\end{aligned}$$

and the boundary conditions:

$$\begin{aligned}z^\mu(s_0) &= -\xi_{[0]}^\nu \left(\frac{\delta \mathcal{K}_\nu^{[0]}}{\delta k_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_\nu^{[0]}}{\delta k_1} k_1 \right), \\ x^\mu(s_0) &= \xi_{[0]}^\nu \left(\frac{\delta \mathcal{K}_\nu^{[0]}}{\delta p_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_\nu^{[0]}}{\delta p_1} p_1 \right), \\ x'^\mu(s_0) &= -\xi_{[0]}^\nu \left(\frac{\delta \mathcal{K}_\nu^{[0]}}{\delta p'_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_\nu^{[0]}}{\delta p'_1} p'_1 \right), \quad x'^\mu(s_1) = \xi_{[1]}^\nu \left(\frac{\delta \mathcal{K}_\nu^{[1]}}{\delta p'_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_\nu^{[1]}}{\delta p'_1} p'_1 \right), \\ x''^\mu(s_1) &= -\xi_{[1]}^\nu \left(\frac{\delta \mathcal{K}_\nu^{[1]}}{\delta p''_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_\nu^{[1]}}{\delta p''_1} p''_1 \right), \\ y^\mu(s_1) &= \xi_{[1]}^\nu \left(\frac{\delta \mathcal{K}_\nu^{[1]}}{\delta q_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_\nu^{[1]}}{\delta q_1} q_1 \right), \\ y'^\mu(s_1) &= -\xi_{[1]}^\nu \left(\frac{\delta \mathcal{K}_\nu^{[1]}}{\delta q'_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_\nu^{[1]}}{\delta q'_1} q'_1 \right).\end{aligned}$$

For the first time in this manuscript we are in this section interested not only in establishing the relativistic properties acquired through our prescription for the choice of boundary terms, but also on the predictions of the formalism for what happens to particles. Evidently here the issue of interest is primarily contained in the dependence of the time of detection at a given detector of simultaneously-emitted particles on the momenta of the particles and on the specific properties of the interactions involved in the analysis. We shall analyze this issue arranging the setup in a way that renders transparent the comparison with the Hamiltonian treatment of free particles reviewed in our Section VI. We start by noticing that for the particle of worldline x^μ , we have

$$x^1(s) = x^1(\bar{s}) + v^1(x^0(s) - x^0(\bar{s})), \quad (101)$$

which in the massless case (and whenever $m/p_1^2 \ll |\ell p_1|$ takes the simple form

$$x^1(s) = x^1(\bar{s}) - \frac{p_1}{|p_1|} (x^0(s) - x^0(\bar{s})). \quad (102)$$

In obtaining (102) we used the on-shell relation

$$p_0 = \sqrt{p_1^2 + m^2} - \frac{\ell}{2} p_1^2,$$

and the fact that for $m/p_1^2 \ll |\ell p_1|$ (consistently again with our choice of conventions, which is such that $v^1 > 0 \implies p^1 < 0$)

$$v^1 = \frac{x^1}{x^0} = -\frac{p_1}{p_0}(1 - \ell p_0 + \frac{\ell p_1^2}{2p_0}) = -\frac{p_1}{|p_1|}.$$

Just as in Sec. VI, we have momentum-independent coordinate speeds for massless particles, so in particular according to Alice's coordinates two massless particles of momenta p_1^s and p_1^h simultaneously emitted at Alice (in Alice's spacetime origin) appear to reach detector Bob simultaneously, apparently establishing a coincidence of detection events. But, as stressed already in Sec. VI, the presence of relative locality evidently requires that in order to establish the dependence of the time of detection on the momentum of the massless particles we must again transform the relevant worldlines to the corresponding description by an observer Bob local to the detection. Let us then return to the two-interaction process of Fig. 8 and take as our hard massless particle of momentum p_1^h the particle in that process which we had originally labeled as having momentum p_1' . For the process of Fig. 8 our description of the transformation from Alice's to Bob's worldlines is

$$\begin{aligned} z_B^0(s) &= z_A^0(s) + b^\mu \{(q \oplus p' \oplus k)_\mu, z^0\} = z_A^0(s) - b^0 - \ell b^1 k_1 \simeq z_A^0(s) - b^0, \\ z_B^1(s) &= z_A^1(s) + b^\mu \{(q \oplus p' \oplus k)_\mu, z^1\} = z_A^1(s) - b^1 - \ell(p_0' + q_0) \simeq z_A^1(s) - b^1 - \ell b^1 p_0', \\ x_B^0(s) &= x_A^0(s) + b^\mu \{(q \oplus p)_\mu, x^0\} = x_A^0(s) - b^0 - \ell b^1 p_1, \\ x_B^1(s) &= x_A^1(s) + b^\mu \{(q \oplus p)_\mu, x^1\} = x_A^1(s) - b^1 - \ell q_0 \simeq x_A^1(s) - b^1, \\ x_B'^0(s) &= x_A'^0(s) + b^\mu \{(q \oplus p' \oplus k)_\mu, x'^0\} = x_A'^0(s) - b^0 - \ell b^1 (k_1 + p_1') \simeq x_A'^0(s) - b^0 - \ell b^1 p_1', \\ x_B'^1(s) &= x_A'^1(s) + b^\mu \{(q \oplus p' \oplus k)_\mu, x'^1\} = x_A'^1(s) - b^1 - \ell q_0 \simeq x_A'^1(s) - b^1, \\ x_B''^0(s) &= x_A''^0(s) + b^\mu \{(p'' \oplus q' \oplus k)_\mu, x''^0\} = x_A''^0(s) - b^0 - \ell b^1 (q_1' + k_1 + p_1'') \simeq x_A''^0(s) - b^0 - \ell b^1 p_1'', \\ x_B''^1(s) &= x_A''^1(s) + b^\mu \{(p'' \oplus q' \oplus k)_\mu, x''^1\} = x_A''^1(s) - b^1, \\ y_B^0(s) &= y_A^0(s) + b^\mu \{(q \oplus p' \oplus k)_\mu, y^0\} = y_A^0(s) - b^0 - \ell b^1 (p_1' + k_1 + q_1) \simeq y_A^0(s) - b^0 - \ell b^1 p_1', \\ y_B^1(s) &= y_A^1(s) + b^\mu \{(q \oplus p' \oplus k)_\mu, y^1\} = y_A^1(s) - b^1, \\ y_B'^0(s) &= y_A'^0(s) + b^\mu \{(p'' \oplus q' \oplus k)_\mu, y'^0\} = y_A'^0(s) - b^0 - \ell b^1 (k_1 + q_1') \simeq y_A'^0(s) - b^0, \\ y_B'^1(s) &= y_A'^1(s) + b^\mu \{(p'' \oplus q' \oplus k)_\mu, y'^1\} = y_A'^1(s) - b^1 - \ell b^1 p_0''. \end{aligned} \tag{103}$$

Using these transformation laws it is easy to recognize that, having dropped the negligible “soft terms” from small momenta, indeed we are obtaining results that are fully consistent with the ones obtained in the hamiltonian description of free particles. To see this explicitly let us consider the situation where, simultaneously to the interaction emitting the hard particle x', p' in Alice origin, we also have the emission of a soft photon x_s, p_s .

And as observer Bob let us take one who is reached in its spacetime origin by the soft photon emitted by Alice. For the event of detection of the hard particle x', p' we take one such that it occurs in Bob's spatial origin.

From a relative-locality perspective the setup we are arranging is such that “Alice is an emitter” (the spatial origin of Alice's coordinate system is an ideally compact, infinitely small, emitter) and “Bob is a detector” (the spatial origin of Bob's coordinate system is an ideally compact, infinitely small, detector). The two worldlines we focus on, a soft and a hard worldline, both originate from Alice's spacetime origin (they are both emitted by Alice, in the spatial origin of Alice's frame of reference, and both at time $t_{\text{Alice}} = 0$) and both end up being detected by Bob, but, while by construction the soft particle reaches Bob's spacetime origin, the time at which the hard particle reaches Bob spatial origin is to be determined through our analysis.

Reasoning as usual at first order in ℓ , it is easy to verify that Bob describes the “interaction coordinate” $\xi_B^{[1]\mu}$ of the interaction at $s = s_1$ as coincident with the $s = s_1$ endpoints of the worldlines $x', p'; x'', p''; q, y; q', y'$:

$$\xi_B^{[1]\mu} = x_B'^\mu(s_1) = x_B''^\mu(s_1) = y_B^\mu(s_1) = y_B'^\mu(s_1). \tag{104}$$

We take into account that there are no relative-locality effects in the description given by Bob whenever an interaction occurs “in the vicinity of Bob”: our leading-order analysis assumes the observatories have sensitivity sufficient to expose manifestation of relativity of locality of order $\ell p_h L$ (where L is the distance from the interaction-event to the origin of the observer and p_h is a “suitably high” momentum), with L set in this case by the distance Alice-Bob, so even a hard-particle interaction which is at a distance d from the origin of Bob will be treated as absolutely local by Bob if $d \ll L$.

According to this both “detection events” are absolutely local for observer Bob: of course this is true for the event of detection of the soft photon x_s, p_s (which we did not even specify since its softness ensures us of its absolute locality) and it is also true for the interaction-event of “detection near Bob” of the hard particle x', p' . Ultimately this allows us to handle the time component of the coordinate fourvector (104) as the actual delay that Bob measures between the two detection times:

$$\Delta t = \xi_B^{[1]0} = x_B'^0(s_1) = x_B''^0(s_1) = y_B^0(s_1) = y_B'^0(s_1). \tag{105}$$

From the equations (103) relative to the worldline x', p' , it follows that

$$x_A'^1(s_1) = x_B'^1(s_1) + b^1 = b^1, \quad (106)$$

from which, considering the worldlines (102), it follows that (assuming indeed $m/(p_1')^2 \ll |\ell p_1'|$) Alice “sees” the $s = s_1$ endpoint of the worldline x', p' at the coordinates

$$x_A'^\mu(s_1) = x_B'^\mu(s_1) + b^\mu = b^\mu = (b, b). \quad (107)$$

And then, from the equations (103) and (105), it follows that Bob measures the delay

$$\Delta t = x_B'^0(s_1) = x_A'^0(s_1) - b^0 - \ell b^1 p_1' = \ell b |p_1'|, \quad (108)$$

in agreement with the result (71) found in the Hamiltonian description. These findings are summarized in Figure 9.

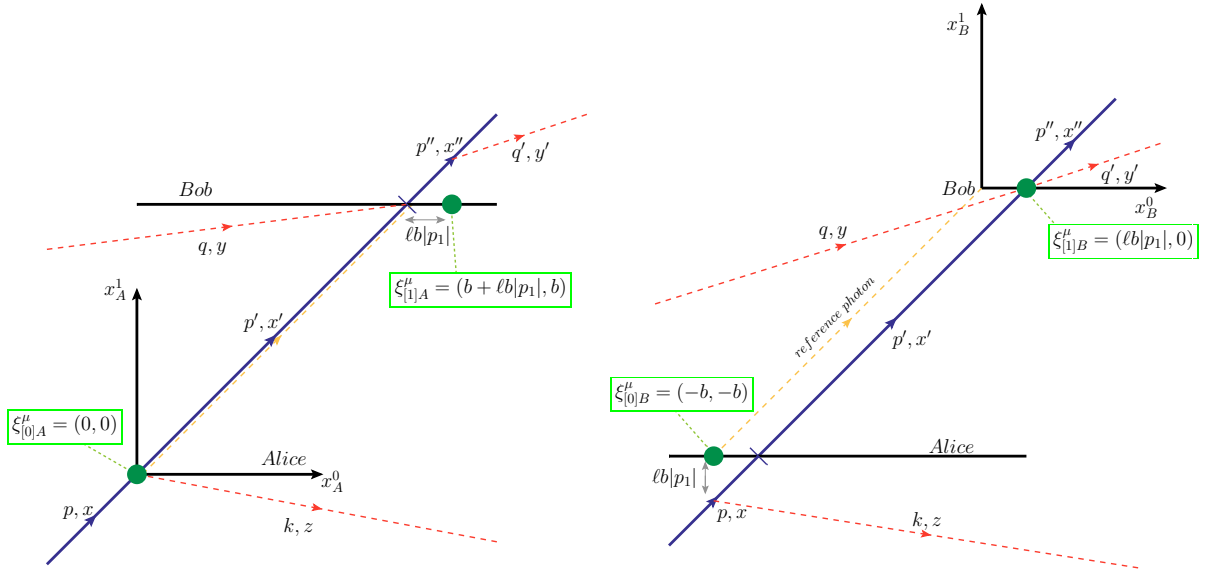


Figure 9. Schematic description of the time delay derived in this subsection. The various agents in the analysis are shown both as described by Alice (left panel) and as described by Bob (right panel). These are spacetime graphs (in a 2D spacetime) showing the actual worldlines of particles. In addition to the two hard interactions we are considering (qualitatively described already in Figure 8), we also show (as the orange-dotted worldline) a soft photon going from Alice’s origin to Bob’s origin. As shown in the figure we have arranged the calculations in this section so that all emissions and detections occur in the spatial origin of either Alice or Bob (but because of the relative locality according to Alice the hard detections at/near Bob would be nonlocal interactions and according to Bob the hard emissions at Alice would be nonlocal processes). We also show, as the bulky green dots, the formal positions of the interaction points, as coded in the formal “interaction coordinates” ξ^μ .

Since we are dealing with a momentum-space with torsion, *i.e.* the momentum composition law is noncommutative, it is interesting to check whether this result establishing agreement with the Hamiltonian description of free particles also holds for other choices of ordering of momenta in the conservation laws (and accordingly in the boundary conditions).

An interesting alternative for the conservation laws and boundary conditions of the process in Figure 8 is the following

$$\begin{aligned} \mathcal{K}_\mu^{[0]}(s_0) &= (p \oplus q)_\mu - (k \oplus p' \oplus q)_\mu = p_\mu - p'_\mu - k_\mu - \ell \delta_\mu^1 (-p_0 q_1 + k_0 p'_1 + k_0 q_1 + p'_0 q_1), \\ \mathcal{K}_\mu^{[1]}(s_1) &= (k \oplus p' \oplus q)_\mu - (k \oplus p'' \oplus q')_\mu = p'_\mu + q_\mu - p''_\mu - q'_\mu - \ell \delta_\mu^1 (-k_0 p'_1 - k_0 q_1 - p'_0 q_1 + k_0 p''_1 + k_0 q'_1 + p''_0 q'_1). \end{aligned} \quad (109)$$

Going from the previous version of the boundary conditions to this one does change several things in the analysis, but it easy to see that it does not change anything about the “free particle” x', p' . With these conservation laws and boundary conditions the relationships between Alice’s worldline x' and Bob’s worldline x' are codified in

$$\begin{aligned} x_B'^0(s) &= x_A'^0(s) + b^\mu \{(k \oplus p' \oplus q)_\mu, x^0\} = x_A'^0(s) - b^0 - \ell b^1 (q_1 + p'_1) \simeq x_A'^0(s) - b^0 - \ell b^1 p'_1, \\ x_B'^1(s) &= x_A'^1(s) + b^\mu \{(k \oplus p' \oplus q)_\mu, x^1\} = x_A'^1(s) - b^1 - \ell b^1 k_0 \simeq x_A'^1(s) - b^1. \end{aligned} \quad (110)$$

And using the equation of motion (102) one easily checks that then the relevant particle reaches Bob's spatial origin at the time

$$\Delta t = x_B'^0(s_1) = x_A^0(s_1) - b^0 - \ell b^1 p_1' = \ell b |p_1'|, \quad (111)$$

in perfect agreement with the result of Eq. (108), which had been obtained with the other choice of ordering of momenta in the conservation laws.

So we find evidence of the fact that the properties of “free particles” (particles exchanging only small fractions of their momentum) are insensitive to the ordering chosen for the law of composition of momenta.

And for what concerns bursts of simultaneously emitted massless particles, such as in a gamma-ray-burst, this derivation predicts differences in times of arrival governed by the formula

$$\Delta t_{arrival} = \ell L |\Delta p_1|,$$

where L is the distance from source to detector (the corresponding translation from observer at the source to observer at the detector has $b^\mu = (L, L)$) and $|\Delta p_1|$ is the difference of momentum among the two massless particles whose arrival times differ by $\Delta t_{arrival}$.

The derivation in this subsection establishes this result for cases where the interaction at the source emitting the particle of interest only involves one hard particle in the in state and one hard particle in the out state (all other particles involved in the interactions being soft).

B. More on observations of distant bursts of massless particles

In the previous subsection, in showing that our proposal (in an appropriate limit) matches the predictions of previous Hamiltonian descriptions of relative locality for free particles, we also showed that, at least for certain types of emission and detection interactions, our proposal predicts time-of-detection delays $\Delta t_{arrival} = \ell L |\Delta p_1|$ between simultaneously-emitted massless particles with momentum difference $|\Delta p_1|$. This is very interesting because, as established in several studies reported over the last decade, such an effect is testable, even if ℓ is as small as the Planck length⁹ (or even one or two orders of magnitude smaller than the Planck length [33–36]).

We derived this time-delay result assuming certain types of emission and detection interactions. But evidently the structure of our formalism is such that it would not be surprising to find that the times of detection depended on the actual emission and detection interactions involved. In this subsection we intend to establish that this is indeed the case, and that the torsion of momentum space plays a crucial role in the relevant analysis.

It suffices to modify the analysis of the previous subsection in rather minor way for us to show that the times of detection of simultaneously emitted particles depend not only on the momenta of the particles but also on the actual nature of the emitting interaction. We find that in order for this to occur there must be at least 3 hard particles in total, among in and out particles of the emission interaction.

As an example of this we consider here explicitly the case of a ultraenergetic particle at rest decaying into two particles, both hard, one of which is the particle detected at our observatory.

As shown in figure we arrange the analysis in exactly the same way as in the previous section, with a tri-valent vertex for the emission interaction and a four-valent vertex for the detection. And the kinematics at the four-valent vertex is left unchanged, involving a soft particle in the in state and a soft particle in the out state. We only change the kinematics of the emission vertex, now assuming that all particles involved are hard.

And we shall again consider two possible choices of conservation-enforcing boundary conditions, suitable for exploring the role of the noncommutativity of the law of composition of momenta. The same two possible choices of conservation-enforcing boundary conditions already considered in the previous subsection.

Let us start again by analyzing as first possibility

$$\begin{aligned} \mathcal{K}_\mu^{[0]}(s_0) &= (q \oplus p)_\mu - (q \oplus p' \oplus k)_\mu = p_\mu - p'_\mu - k_\mu - \ell \delta_\mu^1 (-q_0 p_1 + q_0 p'_1 + q_0 k'_1 + p'_0 k_1), \\ \mathcal{K}_\mu^{[1]}(s_1) &= (q \oplus p' \oplus k)_\mu - (p'' \oplus q' \oplus k)_\mu = q_\mu + p'_\mu - p''_\mu - q'_\mu - \ell \delta_\mu^1 (-q_0 p'_1 - q_0 k_1 - p'_0 k_1 + p''_0 q'_1 + p''_0 k_1 + q'_0 k_1). \end{aligned} \quad (112)$$

The worldlines seen by observer/detector Bob, distant from the emission, that follow from this choice of boundary terms have been already given in Eq. (102). The main difference between the situation in the previous subsection and the situation we are now analyzing is that the “primary”, the particle incoming to the emission interaction, is at rest, with $p_1 = 0$, which also

⁹ We introduced ℓ as a momentum-space property, with dimensions of inverse momentum. When we mention the possibility of ℓ of order the Planck length we are essentially using jargon, a compact way to describe cases where ℓ^{-1} is of order the Planck scale.

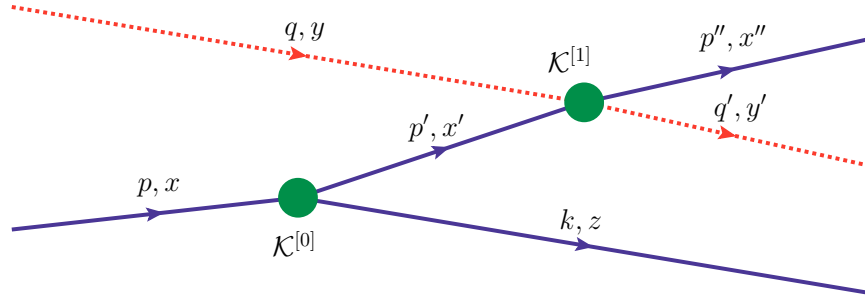


Figure 10. Schematic description of a case where a hard ultrarelativistic particle originating from a hard emission interaction (one hard particle in, two hard particles out) is detected in a soft interaction (only one hard particle in and only one hard particle out). Solid-blue lines are for hard particles, dashed-red lines are for soft particles.

implies that the two outgoing particles of the emission interaction must both be hard. For the worldlines involved in the emission interaction this leads to

$$\begin{aligned}
 x_B^0(s) &= x_A^0(s) + b^\mu \{ (q \oplus p)_\mu, x^0 \} = x_A^0(s) - b^0 - \ell b^1 p_1 = x_A^0(s) - b^0, \\
 x_B^1(s) &= x_A^1(s) + b^\mu \{ (q \oplus p)_\mu, x^1 \} = x_A^1(s) - b^1 - \ell q_0 \simeq x_A^1(s) - b^1, \\
 x_B^0(s) &= x_A^0(s) + b^\mu \{ (q \oplus p' \oplus k)_\mu, x^0 \} = x_A^0(s) - b^0 - \ell b^1 (k_1 + p'_1) = x_A^0(s) - b^0, \\
 x_B^1(s) &= x_A^1(s) + b^\mu \{ (q \oplus p' \oplus k)_\mu, x^1 \} = x_A^1(s) - b^1 - \ell q_0 \simeq x_A^1(s) - b^1, \\
 z_B^0(s) &= z_A^0(s) + b^\mu \{ (q \oplus p' \oplus k)_\mu, z^0 \} = z_A^0(s) - b^0 - \ell b^1 k_1, \\
 z_B^1(s) &= z_A^1(s) + b^\mu \{ (q \oplus p' \oplus k)_\mu, z^1 \} = z_A^1(s) - b^1 - \ell (p'_0 + q_0) \simeq z_A^1(s) - b^1 - \ell b^1 p'_0.
 \end{aligned} \tag{113}$$

And from this one easily sees that the particle p', x' , the particle then detected at Bob, translates classically, without any deformation term. So this time we have that no momentum dependence of the times of arrival is predicted

$$t_{\text{detection}} = x_B^0(s_1) = x_A^0(s_1) - b^0 = 0.$$

Next we show that in this case with the emission interaction involving only hard particles the noncommutativity of the composition law, which had turned out to be uninfluential in the previous subsection, does play a highly non-trivial role. To see this let us consider, as in the previous subsection, the following alternative choice of \mathcal{K} 's for the boundary terms

$$\begin{aligned}
 \mathcal{K}_\mu^{[0]} &= (p \oplus q)_\mu - (k \oplus p' \oplus q)_\mu = p_\mu - p'_\mu - k_\mu - \ell \delta_\mu^1 (-p_0 q_1 + k_0 p'_1 + k_0 q_1 + p'_0 q_1), \\
 \mathcal{K}_\mu^{[1]} &= (k \oplus p' \oplus q)_\mu - (k \oplus p'' \oplus q')_\mu = p'_\mu + q_\mu - p''_\mu - q'_\mu - \ell \delta_\mu^1 (-k_0 p'_1 - k_0 q_1 - p'_0 q_1 + k_0 p''_1 + k_0 q'_1 + p''_0 q'_1).
 \end{aligned} \tag{114}$$

Focusing again on the worldline x', p' detected at Bob we now find

$$\begin{aligned}
 x_B^0(s) &= x_A^0(s) + b^\mu \{ (k \oplus p' \oplus q)_\mu, x^0 \} = x_A^0(s) - b^0 - \ell b^1 (q_1 + p'_1) \simeq x_A^0(s) - b^0 - \ell b^1 p'_1, \\
 x_B^1(s) &= x_A^1(s) + b^\mu \{ (k \oplus p' \oplus q)_\mu, x^1 \} = x_A^1(s) - b^1 - \ell b^1 k_0.
 \end{aligned} \tag{115}$$

And from the equation of motion (102) one now deduces that

$$x_B^1(s) = x_B^0(s) - \ell b^1 (k_0 - p'_1),$$

which in turn implies that the time of detection at Bob of the particle with worldline x', p' is

$$t_{\text{detection}} = x_B^0(s_1) = -\ell b^1 (p'_1 - k_0) = 2\ell b^1 |p'_1|. \tag{116}$$

The dependence of the time of detection on the momentum of the particle being detected is back!
And this dependence is twice as strong as the dependence on momentum found in the previous subsection!

C. Nonmetricity, torsion and time delays

The results we obtained in this subsection are rather striking and deserve to be summarized and discussed in relation with previous related results.

For what concerns times of detection of simultaneously emitted massless particles of momentum p'_1 , emitted from a source at a distance L from the detector we analyzed 3 situations:

(case A) the emission interaction involves only one hard incoming particle and one hard outgoing particle, all other particles in the emission interaction being soft:

the times of arrival have a dependence on momentum governed by

$$t_{\text{detection}} = \ell L |p'_1|$$

and this result is independent of the position occupied by the momentum p'_μ in our noncommutative composition law

(case B) the emission interaction is the decay of a ultra-high-energy particle at rest, involves a total of 3 hard particles, and the momentum p'_μ appears in the composition of momenta to the left of a hard particle:

the times of arrival have no dependence on momentum

$$t_{\text{detection}} = 0$$

(case C) the emission interaction is the decay of a ultra-high-energy particle at rest, involves a total of 3 hard particles, and the momentum p'_μ appears in the composition of momenta to the right of a hard particle:

the times of arrival have the following dependence on momentum

$$t_{\text{detection}} = 2\ell L |p'_1|$$

(twice as large as in the case A).

We obtained this results working with our κ -Poincaré-inspired momentum space, with nonmetricity and torsion.

The fact that it would be possible with such a momentum space to have that simultaneously emitted massless particles are not detected simultaneously at the same detector could be expected on the basis of the analysis reported in Ref. [19], whose main message was that indeed nonmetricity should result in the possibility of having simultaneously emitted massless particles that are not detected simultaneously.

The nature and scopes of our study were such that we could for the first time investigate how the presence of both torsion and nonmetricity could affect these time-of-detection delays. And we evidently found that torsion can have striking effects, effects capable of changing the predicted new effect at order 1, and therefore effects that are as much within reach of ongoing and forthcoming experiments as the pure-nonmetricity (no torsion) effects.

It should be stressed that what we found might even underestimate the significance of the effects of torsion on time delays (at least the effects on time delays of torsion, when also nonmetricity is present). This is because we contemplated a total of only 3 cases for what concerns the kinematics and the conservation laws that are relevant for such an analysis. But even within the confines of our preliminary investigation we found a type of dependence of the time delays, not only on momenta of observed particles but also on interactions that produced them, which had never been encountered before in the literature and would therefore provide a very distinguishing feature of the model of momentum space we here adopted as illustrative example.

IX. AN ALTERNATIVE CHOICE OF SYMPLECTIC STRUCTURE

We showed in the previous section that the class of theories we are studying can have striking manifestation, whose magnitude is extremely small but within the reach of our observatories, at least when they observe particles at rather high energy from very distant astros (*e.g.* multiGeV photons from gamma-ray bursts at redshift greater than 1, as discussed in Ref. [33, 35] and references therein).

It is interesting to ask which of the novel feature of the framework we analyzed should be deemed responsible for these striking and testable novel effects. Specifically: is this due exclusively to the geometry of momentum space, codified in the momentum-space metric and connection? or is there also a role played by the choice we made above of a “ κ -Minkowski inspired” symplectic structure?

In this section we provide evidence of the fact that the choice of symplectic structure is completely irrelevant. The physical content of these relative-locality theories is fully codified in the geometry of momentum space. We argue this by adopting here the same geometry of momentum space as in the previous sections, the “ κ -momentum space”, but changing the symplectic structure. We find that the predictions indeed do not change.

We have verified this for all the applications of “ κ -momentum space” discussed in the previous sections. But let us report here explicitly only one case, the particularly noteworthy case studied in the previous Subsection.VIII B.

We then describe the process in Fig. 10 by the following action

$$\begin{aligned}
\mathcal{S}^{\kappa(2)} = & \int_{s_0}^{+\infty} ds (z^\mu \dot{k}_\mu + \mathcal{N}_k C_\kappa[k]) + \int_{-\infty}^{s_0} ds (x^\mu \dot{p}_\mu + \mathcal{N}_p C_\kappa[p]) \\
& + \int_{s_0}^{s_1} ds (x'^\mu \dot{p}'_\mu + \mathcal{N}_{p'} C_\kappa[p']) + \int_{s_1}^{+\infty} ds (x''^\mu \dot{p}''_\mu + \mathcal{N}_{p''} C_\kappa[p'']) \\
& + \int_{-\infty}^{s_1} ds (y^\mu \dot{q}_\mu + \mathcal{N}_q C_\kappa[q]) + \int_{s_1}^{+\infty} ds (y'^\mu \dot{q}'_\mu + \mathcal{N}_{q'} C_\kappa[q']) \\
& - \xi_{[0]}^\mu \mathcal{K}_\mu^{[0]}(s_0) - \xi_{[1]}^\mu \mathcal{K}_\mu^{[1]}(s_1) ,
\end{aligned} \tag{117}$$

which we take to be identical to the corresponding one written in Subsection.VIII B, with the exception of the evident change from the “ κ -Minkowski symplectic structure” of Subsection.VIII B to the adoption here of a trivial symplectic structure:

$$\{x^1, x^0\} = 0 ,$$

$$\{x^0, p_0\} = 1, \quad \{x^1, p_0\} = 0 , \tag{118}$$

$$\{x^0, p_1\} = 0, \quad \{x^1, p_1\} = 1 . \tag{119}$$

The metric and connection on momentum space are still the ones of the “ κ -momentum space”, and we consider again the same boundary terms also considered in Subsection.VIII B:

$$\begin{aligned}
\mathcal{K}_\mu^{[0]}(s_0) &= (q \oplus p)_\mu - (q \oplus p' \oplus k)_\mu = p_\mu - p'_\mu - k_\mu - \ell \delta_\mu^1 (-q_0 p_1 + q_0 p'_1 + q_0 k'_1 + p'_0 k_1) , \\
\mathcal{K}_\mu^{[1]}(s_1) &= (q \oplus p' \oplus k)_\mu - (p'' \oplus q' \oplus k)_\mu = q_\mu + p'_\mu - p''_\mu - q'_\mu - \ell \delta_\mu^1 (-q_0 p'_1 - q_0 k_1 - p'_0 k_1 + p''_0 q'_1 + p''_0 k_1 + q'_0 k_1) .
\end{aligned} \tag{120}$$

The change of symplectic structure does lead to some changes in the equations of motion

$$\begin{aligned}
\dot{p}_\mu &= 0 , \quad \dot{q}_\mu = 0 , \quad \dot{k}_\mu = 0 , \quad \dot{p}'_\mu = 0 , \quad \dot{p}''_\mu = 0 \\
C_\kappa[p] &= 0 , \quad C_\kappa[q] = 0 , \quad C_\kappa[k] = 0 , \quad C_\kappa[p'] = 0 , \quad C_\kappa[p''] = 0 \\
z^\mu - \mathcal{N}_k \frac{\delta C_\kappa[k]}{\delta k_\mu} &= 0 , \quad y^\mu - \mathcal{N}_q \frac{\delta C_\kappa[q]}{\delta q_\mu} = 0 , \\
y'^\mu - \mathcal{N}_{q'} \frac{\delta C_\kappa[q']}{\delta q'_\mu} &= 0 , \quad x^\mu - \mathcal{N}_p \frac{\delta C_\kappa[p]}{\delta p_\mu} = 0 , \\
x'^\mu - \mathcal{N}_{p'} \frac{\delta C_\kappa[p']}{\delta p'_\mu} &= 0 , \quad x''^\mu - \mathcal{N}_{p''} \frac{\delta C_\kappa[p'']}{\delta p''_\mu} = 0 ,
\end{aligned}$$

and in the boundary conditions:

$$\begin{aligned}
z^\mu(s_0) &= -\xi_{[0]}^\nu \frac{\delta \mathcal{K}_\nu^{[0]}}{\delta k_\mu} \\
x^\mu(s_0) &= \xi_{[0]}^\nu \frac{\delta \mathcal{K}_\nu^{[0]}}{\delta p_\mu} \\
x'^\mu(s_0) &= -\xi_{[0]}^\nu \frac{\delta \mathcal{K}_\nu^{[0]}}{\delta p'_\mu} \quad x'^\mu(s_1) = \xi_{[1]}^\nu \frac{\delta \mathcal{K}_\nu^{[1]}}{\delta p'_\mu} \\
x''^\mu(s_1) &= -\xi_{[1]}^\nu \frac{\delta \mathcal{K}_\nu^{[1]}}{\delta p''_\mu} \\
y^\mu(s_1) &= \xi_{[1]}^\nu \frac{\delta \mathcal{K}_\nu^{[1]}}{\delta q_\mu} \\
y'^\mu(s_1) &= -\xi_{[1]}^\nu \frac{\delta \mathcal{K}_\nu^{[1]}}{\delta q'_\mu} ,
\end{aligned} \tag{121}$$

but in spite of this the predictions remain unchanged.

The changes in the equations of motion appear to take a rather tangible form at intermediate stages of analysis. For example for the particle of worldline x^μ , we have

$$\ddot{x}^\mu = \mathcal{N}_p \frac{\delta C_K[p]}{\delta p_\mu} = \delta_0^\mu \mathcal{N}_p (2p_0 + \ell p_1^2) - 2\delta_1^\mu \mathcal{N}_p (p_1 - \ell p_0 p_1), \quad (122)$$

from which it follows that in the massless case

$$x^1(s) = x^1(\bar{s}) - \left(\frac{p_1}{|p_1|} - \ell p_1 \right) (x^0(s) - x^0(\bar{s})), \quad (123)$$

and this is quite different from the corresponding formula obtained in Subsect. VIII B. But there are other aspects of the analysis which are affected by the change of symplectic structure, and ultimately the predictions of time of detection remain unchanged.

To see this let us focus again on the case in which the “primary”, the particle incoming to the emission interaction, is at rest, with $p_1 = 0$, which also implies that the two outgoing particles of the emission interaction must both be hard. For the worldlines involved in the emission interaction this leads to

$$\begin{aligned} x_B^0(s) &= x_A^0(s) + b^\mu \{(q \oplus p)_\mu, x^0\} = x_A^0(s) - b^0, \\ x_B^1(s) &= x_A^1(s) + b^\mu \{(q \oplus p)_\mu, x^1\} = x_A^1(s) - b^1 - \ell q_0 \simeq x_A^1(s) - b^1, \\ x_B'^0(s) &= x_A'^0(s) + b^\mu \{(q \oplus p' \oplus k)_\mu, x'^0\} = x_A'^0(s) - b^0 - \ell b^1 k_1, \\ x_B'^1(s) &= x_A'^1(s) + b^\mu \{(q \oplus p' \oplus k)_\mu, x'^1\} = x_A'^1(s) - b^1 - \ell q_0 \simeq x_A'^1(s) - b^1, \\ z_B^0(s) &= z_A^0(s) + b^\mu \{(q \oplus p' \oplus k)_\mu, z^0\} = z_A^0(s) - b^0, \\ z_B^1(s) &= z_A^1(s) + b^\mu \{(q \oplus p' \oplus k)_\mu, z^1\} = z_A^1(s) - b^1 - \ell(p'_0 + q_0) \simeq z_A^1(s) - b^1 - \ell b^1 p'_0. \end{aligned} \quad (124)$$

To find the time of detection of the particle x', p' as seen by Bob, who is at Alice coordinates $b^\mu = (b, b)$, we exploit the fact that, assuming that the particle x', p' crosses Bob spatial origin at $s = s_1$,

$$x_A^1(s_1) = x_B^1(s_1) + b^1 = b^1 = b, \quad (125)$$

from which, using the equations of motion (123), rewritten for the particle x', p' as seen by Alice, it follows that

$$x_A^0(s_1) = x_A'^0(s_1) (1 - \ell p'_1) = b - \ell b p'_1. \quad (126)$$

Using again the equations of motion (123), rewritten for the particle x', p' as seen by Bob, together with Eq. (124), we finally find

$$t_{\text{detection}} = x_B^0(s_1) = x_A^0(s_1) - b^0 - \ell b^1 k_1 = -\ell b (k_1 + p'_1) = 0. \quad (127)$$

So we have no momentum dependence of the times of detection exactly for the choice of boundary terms that also in Subsect. VIII B produced the same momentum independence of times of detection.

Let us consider now, as in Subsect. VIII B, the following alternative choice of \mathcal{K} 's for the boundary terms

$$\begin{aligned} \mathcal{K}_\mu^{[0]} &= (p \oplus q)_\mu - (k \oplus p' \oplus q)_\mu = p_\mu - p'_\mu - k_\mu - \ell \delta_\mu^1 (-p_0 q_1 + k_0 p'_1 + k_0 q_1 + p'_0 q_1), \\ \mathcal{K}_\mu^{[1]} &= (k \oplus p' \oplus q)_\mu - (k \oplus p'' \oplus q')_\mu = p'_\mu + q_\mu - p''_\mu - q'_\mu - \ell \delta_\mu^1 (-k_0 p'_1 - k_0 q_1 - p'_0 q_1 + k_0 p''_1 + k_0 q'_1 + p'_0 q'_1). \end{aligned} \quad (128)$$

Focusing again on the worldline x', p' detected at Bob we now find

$$\begin{aligned} x_B^0(s) &= x_A^0(s) + b^\mu \{(k \oplus p' \oplus q)_\mu, x^0\} = x_A^0(s) - b^0 - \ell b^1 q_1 \simeq x_A^0(s) - b^0, \\ x_B^1(s) &= x_A^1(s) + b^\mu \{(k \oplus p' \oplus q)_\mu, x^1\} = x_A^1(s) - b^1 - \ell b^1 k_0. \end{aligned} \quad (129)$$

And from the equation of motion (123) for the particle x', p' as seen by Bob, one now deduces that

$$x_B^1(s) = x_B^0(s) - \ell b^1 (k_0 - p'_1),$$

which in turn implies that the time of detection at Bob of the particle with worldline x', p' is

$$t_{\text{detection}} = x_B^0(s_1) = \ell b^1 (k_0 - p'_1) = 2\ell b^1 |p'_1|. \quad (130)$$

And this once again shows that, in spite of the change of symplectic structure the results of Subsect. VIII B are exactly reproduced: we got dependence on momentum of the times of detection given by $2\ell b^1 |p'_1|$ for exactly the same choice of boundary terms that also in Subsect. VIII B produced dependence on momentum of the times of detection given by $2\ell b^1 |p'_1|$.

X. SUMMARY AND OUTLOOK

The young idea of relative locality of course as a long way to go before taking full shape, but it is encouraging that already some aspects of it which might have been naively perceived as unsurmountable challenges actually turned out, upon closer inspection, to be manifestations of the internal strength of the logical structure of the relative-locality proposal. A striking example of this is the analysis reported in Ref. [20], which addressed some potential challenges for the description of macroscopic bodies. And we feel that we reported in this manuscript a similar accomplishment, by showing that the complication of the nonlinear boundary terms used for enforcing momentum-conservation laws does admit, in spite of its first-look appearance, a fully satisfactory relativistic description of distant observers.

Besides its use for the development of new theories, we should stress in this closing remarks also how awareness of a possible relativity of locality can empower certain analyses: the “ κ -Poincaré phase spaces” which were used extensively in our manuscript were invented long before the recent conceptualization of relative locality (see, *e.g.*, Ref. [37–39]). But the analysis of such κ -Poincaré phase spaces had remained confined for many years at the level of theories of only free particles. No such interacting particle theory had been found, and we now understand why: without the awareness of the possibility of a relativity of locality one could not have introduced the description of interactions that actually works, which we instead here produced with rather little effort. We expect that several such instances may be discovered in future studies, other instances in which an already well-known theoretical framework is shown to host relativity of locality, and is then much better understood once the awareness of the relativity of locality is exploited.

The fact we here settled several issues relevant for translational invariance may be an important step forward for relative locality, but several other such steps need to be taken. In particular, the description of relative locality for distantly boosted observers is at present reasonably well understood only for simple models of relative locality for free particles [15–18]. The description of distantly boosted observers within the relative-locality framework for interacting particles of Refs. [1, 2], which we here adopted, will probably require facing challenges of similar magnitude to (but different nature from) the ones we studied here.

Of course, of primary importance for the relative-locality framework are its phenomenological consequences, and specifically its ability to predict effects that (while surely minute) are within the reach of the sensitivities of ongoing or foreseeable experimental studies. That such opportunities would be found was already clear on the basis of some of the observations reported in Ref. [1], and the results of Ref. [19] connecting nonmetricity to time delays (of the type here considered in Sec. VIII) with encouraging quantitative estimates already gave additional tangibility to these expectations. In a sort of much-welcome corollary to our main work on translational invariance for cases with causally connected interactions, we here exposed, in Sec. VIII, first evidence of some striking (minute but “observably large”) manifestations of the torsion of momentum space, in a case where nonmetricity and torsion are both present (our “ κ -momentum space”). It is perhaps not surprising that the most striking phenomenological consequences of the geometry of momentum space would be found when both nonmetricity and torsion are present. This is likely going to be the “closest target” for the phenomenology of relative-locality momentum spaces, and we feel that it may therefore deserve priority in future investigations of relative locality.

NOTE ADDED

As we were in the final stages of preparation of this manuscript, we became aware of the study reported in Ref. [40], which takes as starting point a characterization of the “ κ -momentum space”, just like here in Sections III and IV we got our analysis started by a characterization of the “ κ -momentum space”. There are some differences in style and focus, but the characterizations of “ κ -momentum space” given here and in Ref. [40] are fully consistent with one another. The issues for the relativistic description of distant observers within the framework of Refs. [1, 2], which are the main objective of the study we reported in this manuscript, (main results in Sections V VII VIII), were not considered in Ref. [40]. Instead Ref. [40] investigates certain issues relevant for the implementation of κ -Poincaré boosts in a relative-locality setting of the type advocated in Refs. [1, 2]. We expect that the interplay between the relativistic description of distant observers we provided here and the properties of κ -Poincaré boosts highlighted in Ref. [40] could make for an entertaining future project.

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Appendix A: Equations of motion from the action of Subsection VII C

In this appendix we write down explicitly the equations of motion, constraints and boundary conditions that are relevant for the discussion given in Subsec. VII C.

These are the equations of motion and boundary conditions that follow from the action

$$\begin{aligned}
\mathcal{S}^{\kappa(2')} = & \int_{-\infty}^{s_0} ds \left(z^\mu \dot{k}_\mu - \ell z^1 k_1 \dot{k}_0 + \mathcal{N}_k C_\kappa[k] \right) + \int_{s_0}^{s_1} ds \left(x^\mu \dot{p}_\mu - \ell x^1 p_1 \dot{p}_0 + \mathcal{N}_p C_\kappa[p] \right) \\
& + \int_{s_0}^{s_1} ds \left(y^\mu \dot{q}_\mu - \ell y^1 q_1 \dot{q}_0 + \mathcal{N}_q C_\kappa[q] \right) + \int_{s_1}^{\infty} ds \left(y_\star^\mu \dot{q}_\mu^\star - \ell y_\star^1 q_1^\star \dot{q}_0^\star + \mathcal{N}_{q^\star} C_\kappa[q^\star] \right) \\
& + \int_{s_1}^{\infty} ds \left(x'^\mu \dot{p}'_\mu - \ell x'^1 p'_1 \dot{p}'_0 + \mathcal{N}_{p'} C_\kappa[p'] \right) + \int_{s_1}^{\infty} ds \left(x''^\mu \dot{p}''_\mu - \ell x''^1 p''_1 \dot{p}''_0 + \mathcal{N}_{p''} C_\kappa[p''] \right) \\
& - \xi_{[0]}^\mu \mathcal{K}_\mu^{[0]}(s_0) - \xi_{[1]}^\mu \mathcal{K}_\mu^{[1]}(s_1) - \xi_{[1']}^\mu \mathcal{K}_\mu^{[1']}(s_1) ,
\end{aligned} \tag{A1}$$

and one easily finds

$$\begin{aligned}
\dot{p}_\mu = 0, \quad \dot{q}_\mu = 0, \quad \dot{k}_\mu = 0, \quad \dot{p}'_\mu = 0, \quad \dot{p}''_\mu = 0, \quad \dot{q}^\star_\mu = 0, \\
C_\kappa[p] = 0, \quad C_\kappa[q] = 0, \quad C_\kappa[k] = 0, \quad C_\kappa[p'] = 0, \quad C_\kappa[p''] = 0, \quad C_\kappa[q^\star] = 0, \\
\mathcal{K}_\mu^{[0]} = 0, \quad \mathcal{K}_\mu^{[1]} = 0, \quad \mathcal{K}_\mu^{[1']} = 0.
\end{aligned} \tag{A2}$$

$$\begin{aligned}
\ddot{x}^\mu &= \mathcal{N}_p \left(\frac{\delta C_\kappa[p]}{\delta p_\mu} + \ell \delta_0^\mu \frac{\delta C_\kappa[p]}{\delta p_1} p_1 \right) = \delta_0^\mu \mathcal{N}_p (2p_0 - \ell p_1^2) - 2\delta_1^\mu \mathcal{N}_p (p_1 - \ell p_0 p_1) , \\
\ddot{y}^\mu &= \mathcal{N}_q \left(\frac{\delta C_\kappa[q]}{\delta q_\mu} + \ell \delta_0^\mu \frac{\delta C_\kappa[q]}{\delta q_1} q_1 \right) = \delta_0^\mu \mathcal{N}_q (2q_0 - \ell q_1^2) - 2\delta_1^\mu \mathcal{N}_q (q_1 - \ell q_0 q_1) , \\
\ddot{z}^\mu &= \mathcal{N}_k \left(\frac{\delta C_\kappa[k]}{\delta k_\mu} + \ell \delta_0^\mu \frac{\delta C_\kappa[k]}{\delta k_1} k_1 \right) = \delta_0^\mu \mathcal{N}_k (2k_0 - \ell k_1^2) - 2\delta_1^\mu \mathcal{N}_k (k_1 - \ell k_0 k_1) , \\
\ddot{x}'^\mu &= \mathcal{N}_{p'} \left(\frac{\delta C_\kappa[p']}{\delta p'_\mu} + \ell \delta_0^\mu \frac{\delta C_\kappa[p']}{\delta p'_1} p'_1 \right) = \delta_0^\mu \mathcal{N}_{p'} (2p'_0 - \ell p_1'^2) - 2\delta_1^\mu \mathcal{N}_{p'} (p'_1 - \ell p'_0 p'_1) , \\
\ddot{x}''^\mu &= \mathcal{N}_{p''} \left(\frac{\delta C_\kappa[p'']}{\delta p''_\mu} + \ell \delta_0^\mu \frac{\delta C_\kappa[p'']}{\delta p''_1} p''_1 \right) = \delta_0^\mu \mathcal{N}_{p''} (2p''_0 - \ell p_1''^2) - 2\delta_1^\mu \mathcal{N}_{p''} (p''_1 - \ell p''_0 p''_1) , \\
\ddot{y}^{\star\mu} &= \mathcal{N}_{q^\star} \left(\frac{\delta C_\kappa[q^\star]}{\delta q_\mu^\star} + \ell \delta_0^\mu \frac{\delta C_\kappa[q^\star]}{\delta q_1^\star} q_1^\star \right) = \delta_0^\mu \mathcal{N}_{q^\star} (2q_0^\star - \ell q_1^{\star 2}) - 2\delta_1^\mu \mathcal{N}_{q^\star} (q_1^\star - \ell q_0^\star q_1^\star) .
\end{aligned} \tag{A3}$$

It is easy to recognize that these are the same equations of motion that one also obtains from the action $\mathcal{S}^{\kappa(2)}$ (of Subsec. VII B), up to splitting fictitiously (as suggested by the drawing in Fig. 6) the worldline y^μ, q_μ into two perfectly-matching pieces of worldline, a piece labeled again y^μ, q_μ and a piece labeled y_\star^μ, q_μ^\star .

Similarly one has that from $\mathcal{S}^{\kappa(2')}$ it follows that the conditions at the $s = s_0$ and $s = s_1$ boundaries are

$$\begin{aligned}
z^\mu(s_0) &= \xi_{[0]}^\nu \left(\frac{\delta \mathcal{K}_\nu^{[0]}}{\delta k_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_\nu^{[0]}}{\delta k_1} k_1 \right) = \xi_{[0]}^\mu + \ell \delta_0^\mu \xi_{[0]}^1 k_1 , \\
x^\mu(s_0) &= -\xi_{[0]}^\nu \left(\frac{\delta \mathcal{K}_\nu^{[0]}}{\delta p_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_\nu^{[0]}}{\delta p_1} p_1 \right) = \xi_{[0]}^\mu + \ell \delta_0^\mu \xi_{[0]}^1 (p_1 + q_1) , \\
x^\mu(s_1) &= \xi_{[1]}^\nu \left(\frac{\delta \mathcal{K}_\nu^{[1]}}{\delta p_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_\nu^{[1]}}{\delta p_1} p_1 \right) = \xi_{[1]}^\mu + \ell \delta_0^\mu \xi_{[1]}^1 (p_1 + q_1) , \\
y^\mu(s_0) &= -\xi_{[0]}^\nu \left(\frac{\delta \mathcal{K}_\nu^{[0]}}{\delta q_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_\nu^{[0]}}{\delta q_1} q_1 \right) = \xi_{[0]}^\mu + \ell \delta_0^\mu \xi_{[0]}^1 q_1 + \ell \delta_1^\mu \xi_{[0]}^1 p_0 , \\
y^\mu(s_1) &= \xi_{[1']}^\nu \left(\frac{\delta \mathcal{K}_\nu^{[1']}}{\delta q_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_\nu^{[1']}}{\delta q_1} q_1 \right) = \xi_{[1']}^\mu + \ell \delta_0^\mu \xi_{[1']}^1 q_1 + \ell \delta_1^\mu \xi_{[1']}^1 p_0 , \\
x'^\mu(s_1) &= -\xi_{[1]}^\nu \left(\frac{\delta \mathcal{K}_\nu^{[1]}}{\delta p'_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_\nu^{[1]}}{\delta p'_1} p'_1 \right) = \xi_{[1]}^\mu + \ell \delta_0^\mu \xi_{[1]}^1 (p'_1 + p''_1 + q_1) , \\
x''^\mu(s_1) &= -\xi_{[1]}^\nu \left(\frac{\delta \mathcal{K}_\nu^{[1]}}{\delta p''_\mu} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_\nu^{[1]}}{\delta p''_1} p''_1 \right) = \xi_{[1]}^\mu + \ell \delta_0^\mu \xi_{[1]}^1 (p''_1 + q_1) + \ell \delta_1^\mu \xi_{[1]}^1 p'_0 , \\
y^{*\mu}(s_1) &= -\xi_{[1']}^\nu \left(\frac{\delta \mathcal{K}_\nu^{[1']}}{\delta q_\mu^*} + \ell \delta_0^\mu \frac{\delta \mathcal{K}_\nu^{[1']}}{\delta q_1^*} q_1^* \right) = \xi_{[1']}^\mu + \ell \delta_0^\mu \xi_{[1']}^1 q_1^* + \ell \delta_1^\mu \xi_{[1']}^1 p_0 ,
\end{aligned} \tag{A4}$$

and these essentially reproduce the boundary conditions obtained from $\mathcal{S}^{\kappa(2)}$ in Subsec. VII B, up to boundary conditions that ensure the perfect matching of the two “pieces of worldline” y^μ , q_μ and y_\star^μ , q_μ^* into a single worldline.

Appendix B: Translation transformations of the action in Subsection VII D

We here study the effect of our translation transformations, given in (98), on the action $\mathcal{S}^{(3conn)}$ discussed in Subsection VII D.

The action $\mathcal{S}^{(3conn)}$ includes quite a few terms, and we find it to be convenient to organize them in a way that helps one to keep track of all contributions. A useful expedient is to rewrite the integrations concerning finite worldlines, of the kind $\int_{s_0}^{s_1}$, splitting them into two semi-infinite integrals:

$$\int_{s_0}^{s_1} = \int_{s_0}^{\infty} - \int_{s_1}^{\infty} . \tag{B1}$$

We can then rewrite our action as a sum of semi-infinite integrals and in particular we can opt for the following form

$$\mathcal{S}^{(3conn)} = \mathcal{S}^{(3conn)[s_0]} + \mathcal{S}^{(3conn)[s_1]} + \mathcal{S}^{(3conn)[s_2]} + \mathcal{S}^{(3conn)[s_3]} , \tag{B2}$$

where in $\mathcal{S}^{[s]}$ we include all semi-infinite integrals with a boundary at $s = \bar{s}$.

Using manipulations we already discussed in Subsec. VII B one easily finds that the action for observer Bob can be written as

$$\begin{aligned}
\mathcal{S}_B^{(3conn)} &= \mathcal{S}_A^{(3conn)[s_0]} + \mathcal{S}_A^{(3conn)[s_1]} - \int_{s_2}^{+\infty} ds \left(x^\mu \dot{p}_\mu - \ell x^1 p_1 \dot{p}_0 + \mathcal{N}_p C[p] \right) + \int_{s_2}^{+\infty} ds \left(x'^\mu \dot{p}'_\mu - \ell x'^1 p'_1 \dot{p}'_0 + \mathcal{N}_{p'} C[p'] \right) \\
&\quad + \int_{s_2}^{+\infty} ds \left(x''^\mu \dot{p}''_\mu - \ell x''^1 p''_1 \dot{p}''_0 + \mathcal{N}_{p''} C[p''] \right) \\
&\quad - \int_{s_3}^{+\infty} ds \left(y^\mu \dot{q}_\mu - \ell y^1 q_1 \dot{q}_0 + \mathcal{N}_q C[q] \right) + \int_{s_3}^{+\infty} ds \left(y'^\mu \dot{q}'_\mu - \ell y'^1 q'_1 \dot{q}'_0 + \mathcal{N}_{q'} C[q'] \right) \\
&\quad + \int_{s_3}^{+\infty} ds \left(y''^\mu \dot{q}''_\mu - \ell y''^1 q''_1 \dot{q}''_0 + \mathcal{N}_{q''} C[q''] \right) \\
&\quad - (\Delta \xi_{[0]}^\mu + b^\mu) \mathcal{K}_\mu^{[0]}(s_0) - (\Delta \xi_{[1]}^\mu + b^\mu) \mathcal{K}_\mu^{[1]}(s_1) - \xi_{[2]B}^\mu \mathcal{K}_\mu^{[2]}(s_2) - \xi_{[3]B}^\mu \mathcal{K}_\mu^{[3]}(s_3) \\
&\quad - \int_{s_1}^{+\infty} ds \left(\ell b^1 \mathcal{K}_0^{[1]} \dot{k}_1'' \right) ,
\end{aligned} \tag{B3}$$

where $\mathcal{S}_A^{(3conn)[s_0]}$ and $\mathcal{S}_A^{(3conn)[s_1]}$ are described from Alice's perspective, and we already showed above that the term $\int_{s_1}^{+\infty} ds \left(\ell b^1 \mathcal{K}_0^{[1]} \dot{k}_1'' \right)$ can be dropped since it does not contribute to the equations of motion and the boundary conditions.

Next we can focus on the contributions from integrals with a boundary at $s = s_2$, finding that

$$\begin{aligned} \mathcal{S}_B^{(3conn)[s_2]} = & - \int_{s_2}^{+\infty} ds \left(x^\mu \dot{p}_\mu - \ell x^1 p_1 \dot{p}_0 + \mathcal{N}_p C[p] \right) + \int_{s_2}^{+\infty} ds \left(x'^\mu \dot{p}'_\mu - \ell x'^1 p'_1 \dot{p}'_0 + \mathcal{N}_{p'} C[p'] \right) \\ & + \int_{s_2}^{+\infty} ds \left(x''^\mu \dot{p}''_\mu - \ell x''^1 p''_1 \dot{p}''_0 + \mathcal{N}_{p''} C[p''] \right) \\ & - \xi_{[2]B}^\mu \mathcal{K}_\mu^{[2]}(s_2) , \end{aligned} \quad (B4)$$

after applying our notion of translation transformation, can be written as

$$\begin{aligned} \mathcal{S}_B^{(3conn)[s_2]} = & \mathcal{S}_A^{(3conn)[s_2]} - \int_{s_2}^{+\infty} ds \left(-b^\mu \dot{p}_\mu - \ell b^1 (q_1 + k_1'') \dot{p}_0 \right) + \int_{s_2}^{+\infty} ds \left(-b^\mu \dot{p}'_\mu - \ell b^1 (p'_1 + q_1 + k_1'') \dot{p}'_0 \right) \\ & + \int_{s_2}^{+\infty} ds \left(-b^\mu \dot{p}''_\mu - \ell b^1 (q_1 + k_1'') \dot{p}''_0 - \ell b^1 p'_0 \dot{p}''_1 \right) - \Delta \xi_{[2]}^\mu \mathcal{K}_\mu^{[2]}(s_2) \\ = & \mathcal{S}_A^{(3conn)[s_2]} + \int_{s_2}^{+\infty} ds \frac{d}{ds} (b^\mu \mathcal{K}_\mu^{[2]}) - \Delta \xi_{[2]}^\mu \mathcal{K}_\mu^{[2]}(s_2) \\ & - \int_{s_2}^{+\infty} ds \left(\ell b^1 \mathcal{K}_0^{[2]} (\dot{q}_1 + k_1'') \right) . \end{aligned} \quad (B5)$$

The last term is again of the type that does not contribute to equations of motion and boundary conditions (once the conservation laws are enforced) and can therefore be dropped, while for the contribution

$$\int_{s_2}^{+\infty} ds \frac{d}{ds} (b^\mu \mathcal{K}_\mu^{[2]}) - \Delta \xi_{[2]}^\mu \mathcal{K}_\mu^{[2]}(s_2) ,$$

we find that, since $\Delta \xi_{[2]}^\mu = \xi_{[2]B}^\mu - \xi_{[2]A}^\mu = -b^\mu$, it only produces a boundary term at $s = +\infty$:

$$\int_{s_2}^{+\infty} ds \frac{d}{ds} (b^\mu \mathcal{K}_\mu^{[2]}) - \Delta \xi_{[2]}^\mu \mathcal{K}_\mu^{[2]} = b^\mu \mathcal{K}_\mu^{[2]}(\infty) - (\Delta \xi_{[2]}^\mu + b^\mu) \mathcal{K}_\mu^{[2]}(s_2) = b^\mu \mathcal{K}_\mu^{[2]}(\infty) ,$$

and the boundary term at infinity does not contribute to the physics since momenta are not varied at $\pm\infty$.

Similarly for contributions to the action by semi-infinite integrals with a boundary at $s = s_3$ we have

$$\begin{aligned} \mathcal{S}_B^{(3conn)[3]} = & - \int_{s_3}^{+\infty} ds \left(y^\mu \dot{q}_\mu - \ell y^1 q_1 \dot{q}_0 + \mathcal{N}_q C[q] \right) + \int_{s_3}^{+\infty} ds \left(y'^\mu \dot{q}'_\mu - \ell y'^1 q'_1 \dot{q}'_0 + \mathcal{N}_{q'} C[q'] \right) \\ & + \int_{s_3}^{+\infty} ds \left(y''^\mu \dot{q}''_\mu - \ell y''^1 q''_1 \dot{q}''_0 + \mathcal{N}_{q''} C[q''] \right) \\ & - \xi_{[3]B}^\mu \mathcal{K}_\mu^{[3]}(s_3) , \end{aligned} \quad (B6)$$

which, using again properties of our translation transformations, can be written as

$$\begin{aligned} \mathcal{S}_B^{(3conn)[3]} = & \mathcal{S}_A^{(3conn)[3]} - \int_{s_3}^{+\infty} ds \left(-b^\mu \dot{q}_\mu - \ell b^1 k_1'' \dot{q}_0 - \ell b^1 (p'_0 + p''_0) \dot{q}_1 \right) + \int_{s_3}^{+\infty} ds \left(-b^\mu \dot{q}'_\mu - \ell b^1 (q'_1 + k_1'') \dot{q}'_0 - \ell b^1 (p'_0 + p''_0) \dot{q}'_1 \right) \\ & + \int_{s_3}^{+\infty} ds \left(-b^\mu \dot{q}''_\mu - \ell b^1 k_1'' \dot{q}''_0 - \ell b^1 (p'_0 + p''_0 + q'_0) \dot{q}''_1 \right) - \Delta \xi_{[3]}^\mu \mathcal{K}_\mu^{[3]}(s_3) \\ = & \mathcal{S}_A^{(3conn)[3]} + \int_{s_3}^{+\infty} ds \frac{d}{ds} (b^\mu \mathcal{K}_\mu^{[3]}) - \Delta \xi_{[3]}^\mu \mathcal{K}_\mu^{[3]}(s_3) \\ & - \int_{s_3}^{+\infty} ds \left(\ell b^1 \mathcal{K}_0^{[3]} \dot{k}_1'' + \ell b^1 \mathcal{K}_1^{[3]} (\dot{p}'_0 + \dot{p}''_0) \right) . \end{aligned} \quad (B7)$$

And again we notice that $\int_{s_3}^{+\infty} ds \left(\ell b^1 \mathcal{K}_0^{[3]} \dot{k}_1'' + \ell b^1 \mathcal{K}_1^{[3]} (\dot{p}'_0 + \dot{p}''_0) \right)$ can be dropped since it does not contribute to equations of motion and boundary terms, and as before we have that, since $\xi_{[3]B}^\mu = \xi_{[3]A}^\mu - b^\mu$,

$$\int_{s_3}^{+\infty} ds \frac{d}{ds} (b^\mu \mathcal{K}_\mu^{[3]}) - \Delta \xi_{[3]}^\mu \mathcal{K}_\mu^{[3]}(s_3) = b^\mu \mathcal{K}_\mu^{[3]}(\infty) - (\Delta \xi_{[3]}^\mu + b^\mu) \mathcal{K}_\mu^{[3]}(s_3) = b^\mu \mathcal{K}_\mu^{[3]}(\infty) ,$$

i.e. once again the only left-over piece is a boundary term at infinity that can be safely dropped.

Combining these observations we conclude that both Alice and Bob can use the same action principle to characterize the equations of motion and boundary conditions for the chain of interactions we analyzed in Subsec. VIID.

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